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How Do We Know That's the MINIMUM?

Reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many contexts.

—NCTM, *Principles and Standards for School Mathematics*, p. 56

Combinatorial optimization, which encompasses a variety of minimum-maximum problems, such as finding the shortest route between two cities, is a topic that is especially rich in opportunities to introduce mathematical reasoning. It is an active area of mathematics with connections to real-world problems and is broadly accessible and engaging. In this article, I show how I use two well-known examples, the Map Coloring problem and the Traveling Salesperson problem, to teach logical concepts and methods of proof.

The scenarios described are based on a mathematics course for first-year college liberal arts students. I have adapted this material successfully for in-service teachers of grades 7–12 as well as for undergraduate mathematics majors.

Many of the problems used here are based on those used in the NSF-sponsored "Leadership Program in Discrete Mathematics" (Rosenstein and DeBellis 1997), in which I participated as an instructor. Those problems in turn were adapted from a number of sources. I also use COMAP resources, such as *For All Practical Purposes* (COMAP 2003), *The Mathematician's Coloring Book (*Francis 1989), and *Problem Solving Using Graphs* (Cozzens and Porter 1987), as well as standard textbooks such as *Discrete Mathematics and Its Applications* (Rosen 1999). Additional resources for using combinatorial optimization in mathematics teaching can be found in *Discrete Mathematics across the Curriculum (*NCTM 1991), *Discrete Mathematics in the Schools* (Rosenstein, Franzblau, and Roberts 1997), and *Principles and Standards for School Mathematics* (NCTM 2000).

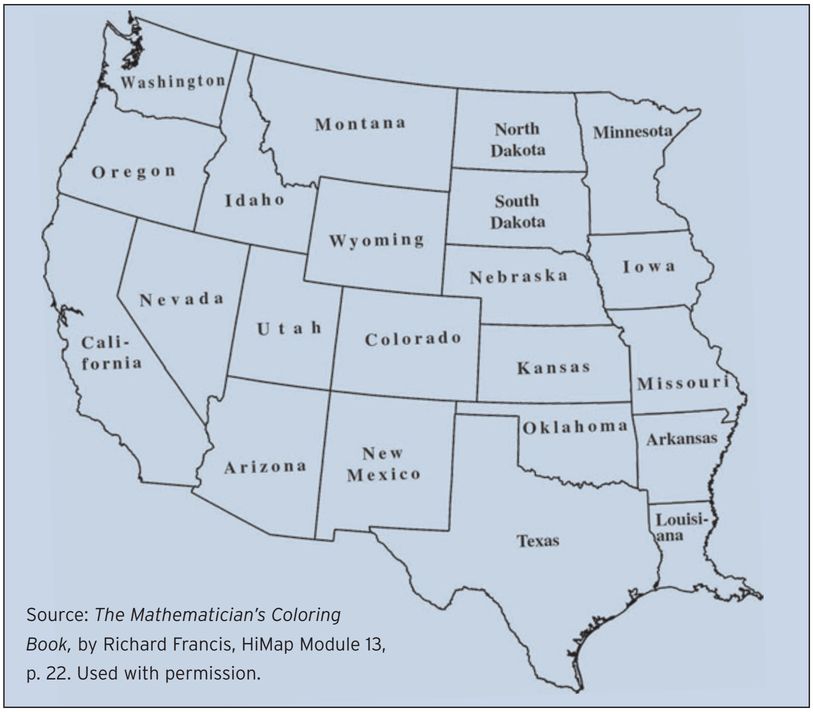
MAP COLORING: SUFFICIENT VS. NECESSARY

The first problem I discuss is inspired by the four-color theorem, which says that any planar map can be colored with four or fewer colors, without the same color touching. The problem of coloring a map has a long history (Saaty 1972). The first proof that four colors are sufficient generated substantial controversy because it required checking hundreds of cases by computer. See, for example, Rosen 1999, p. 512, or Thomas 1998.

To introduce map coloring, I show a map of the western United States. I use the map from *The Mathematician's Coloring Book* (Francis 1989, reproduced in **figure 1**), but any outline map with the state names included would work. I present the problem of using colors to distinguish states that share a border. The problem is most interesting when pairs such as Utah and New Mexico (or Arizona and Colorado) are allowed to have the same color, so I ask the class to agree to this condition.

I assign students to groups of three or four and give each group a copy of the map along with a set of removable colored dots. Students are challenged to find the minimum number of colors needed.

Eventually, the students conclude that they have to use four colors. I ask them why it is impossible to use three colors. The following imaginary dialogue, a composite of many class discussions, illustrates how I try to lead students from intuitive reasoning to formulating and testing specific conjectures.



Source: *The Mathematician's Coloring Book,* by Richard Francis, HiMap Module 13, p. 22. Used with permission.

Fig. 1 Map of the United States west of the Mississippi River

Student: Our group tried over and over to color the map, and we always got four colors.
Teacher: How do you know that another group won't find a way to use three colors? How do you know that *you* won't find a way to use three colors tomorrow?
S: There are just too many connections—you always get stuck.
T: Can you show us exactly where on the map you get stuck?
S: How about Wyoming? It touches six other states.
S: I know—if a state is touching at least three others, then four colors are needed.
T: Let's look at these ideas more closely (writing statement on the board). This is an example of a *conjecture*—the mathematical term for a reasonable guess. It makes sense, but we cannot use it unless it's *always* true. Now let's look at Wyoming and its neighbors carefully (on a transparency). Are four colors required for these seven states?
S: Oh (coloring map on transparency)—I guess you only need three colors. You can color Wyoming blue, then alternate green and yellow for the others.
T: We've just shown that this conjecture is false by giving a counterexample: Wyoming has more than three neighbors, but only three colors are needed for Wyoming plus its neighbors. We must look further to explain why it is impossible to color the whole map with three colors.
S: I just noticed that there are twenty-two states. That's an even number. How can you color an even number of states with an odd number of colors?
T: That's another good conjecture to check (writing statement on the board). Everyone try this now: Try to design your own small map that has an even number of states but can be colored with three colors.
T: (after students show counterexamples) So now we know that this conjecture is also false because we found a counterexample. The number of states doesn't seem to matter.

After a while, a few students realize that the subregion consisting of Nevada and its five

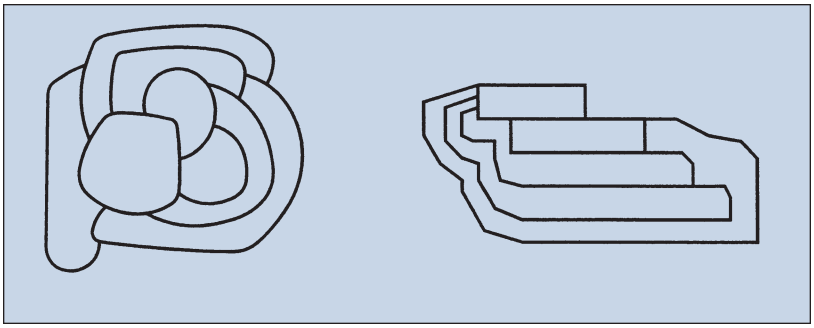


Fig. 2 Two maps that seem to require five colors but can be colored with four colors

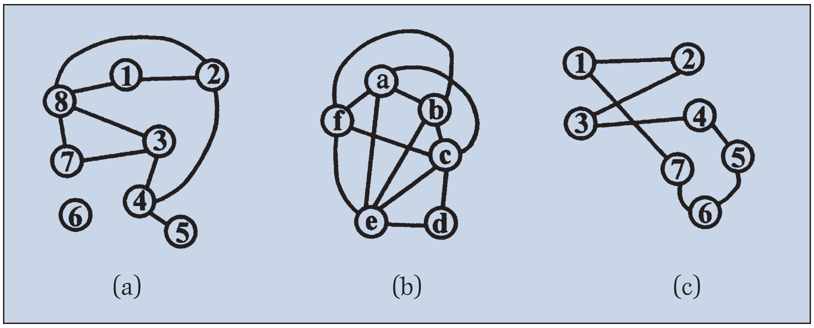


Fig. 3 Examples of graphs. Circles are vertices and lines joining circles are edges. **(a)** (Disconnected) Graph with eight vertices. **(b)** Vertices *a, b, c, e, f* form a complete five-graph, or five-clique: Every pair of vertices is joined. **(c)** A seven-cycle: Each pair of consecutively numbered vertices is joined.

surrounding states requires four colors (see **fig. 1**). At least one student is often able to give a valid argument for this fact. For example, "Idaho must be different from Nevada, and Utah must be different from both, so color these three states red, green, and blue. Since we are trying to avoid a fourth color, Oregon must be blue and Arizona must be red. Then California touches red, green, and blue, so it must be a fourth color, say, orange."

At this point, students have been introduced implicitly to several important logical concepts. First, a mathematical statement is *true* if and only if it holds in every instance; one counterexample is enough to show that a general statement is false. Moreover, proving that a condition is *sufficient* is different from proving that it is *necessary.* Here, four colors are sufficient if there exists a valid four-coloring, that is, if *you* can color this map with four colors; four colors are necessary if every valid coloring has at least four colors, that is, *anyone* who tries to color this map will be forced to use at least four colors.

I reinforce the difference between *sufficient* and *necessary* in several ways. For example, before introducing the four-color theorem, I challenge students to create a map that requires five colors. **Figure 2** shows two maps based on student designs. I display several of these in class, and the students discover that even though *they* used five colors, others may need only four colors: so five colors are sufficient, but not necessary.

GRAPH COLORING: BASIC PROOF METHODS

Map coloring is one of a large set of conflict problems that can be modeled by the more abstract problem of graph coloring, which is useful for introducing both direct proof and proof by contradiction. A graph is a set of *vertices,* such that some pairs of vertices are joined by *edges;* the vertices are usually drawn as a set of dots or circles and the edges as straight or curved lines, as illustrated in **figure 3.** A valid coloring of a graph is an assignment of colors to the vertices so that adjacent vertices (those joined by edges) have different colors. The Graph Coloring problem asks for a minimum coloring, a valid coloring that uses as few colors as possible.

I ask students to find a minimum coloring for the graphs shown in **figure 3** and to prove their answers, using the same ideas they used for maps. I remind students that such a proof requires two separate arguments: To prove that n colors are sufficient, they only have to give a valid coloring with n colors; to prove that n colors are necessary, however, they must show that n colors are forced, no matter what coloring method is used.

For example, the graph in **figure 3(a)** requires three colors, since vertex 3 and vertex 7 must be different from each other, and vertex 8 must be different from both. Using a similar direct proof, one can show that the graph in **figure 3(b)** requires five colors. One can also argue that every pair of vertices among *a, b, c, e,* and *f* is joined by an edge, so no pair can have the same color. Thus, since a separate color is needed for each vertex, five colors are necessary.

The graph in **figure 3(c)** is often called a seven-cycle. To show that the graph requires three colors, it is most natural to use a proof by contradiction, such as the following.

> Suppose that only two colors are available. Let's use green and purple.
>
> Vertex 1 can have either color. If vertex 1 is green, vertex 2 *must* be purple, since it is adjacent to vertex 1. Using the same argument, vertices 1 to 6 *must* alternate green and purple. However, we get stuck at vertex 7, since 1 is green and 6 is purple. Therefore, two colors are impossible. In other words, we need a third color.

Another strategy is to show first that at most three vertices can have any given color. Then, if only two colors were available, only $2 \times 3 = 6$ vertices could be colored.

GRAPH COLORING: BUILDING PROOFS FROM PREVIOUS RESULTS

To introduce the concepts of generalization and of using previous results to prove new results, I ask students to look for patterns in coloring families of graphs and to prove their answers.

One example I use is the family of *cycles,* shown in **figure 4.** For *even cycles* (cycles with an even number of vertices), proving that two colors are necessary and sufficient is straightforward. For the *odd cycles,* three colors are necessary and sufficient. I work with students to generalize their proofs for the seven-cycle discussed in the previous section. See **figure 3(c)**.

Closely related to the cycles is the family of *wheels,* shown in **figure 5.** Students soon realize that they can justify their formulas by using the results for cycles, which we have already proved. I reinforce this idea by assigning graphs to color that contain cycles or wheels as subgraphs.

To assess understanding, I also assign graphs such as the one in **figure 6.** Although the graph appears to contain a seven-wheel, making four colors necessary, edge (*a, h*) is missing, and the graph can actually be colored with three colors.

I also show how to model map coloring as graph coloring, as illustrated in **figure 7.** Looking back at the subgraph formed by Nevada and its neighboring states (**fig. 1**), we can now say that the map of the western United States requires four colors because its graph contains a five-wheel.

To follow up, students use graph coloring to model other conflict problems, such as scheduling meeting times for groups that share members (NCTM 2000, p. 285), assigning chemicals to train cars to avoid harmful reactions (Picker 1997), or selecting frequencies for radio stations to avoid interference (Hart 1997).

TRAVELING SALESPERSON: PROOF BY ENUMERATION

The last fundamental proof method I introduce at this level is enumeration, or systematically check-

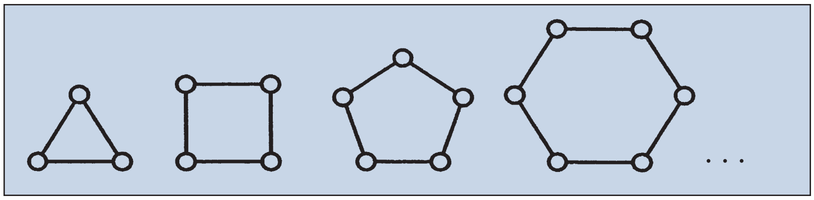


Fig. 4 The cycle family, k = 3, 4, 5, 6. A k-cycle has k vertices; a cycle is even if k is even and odd if k is odd.

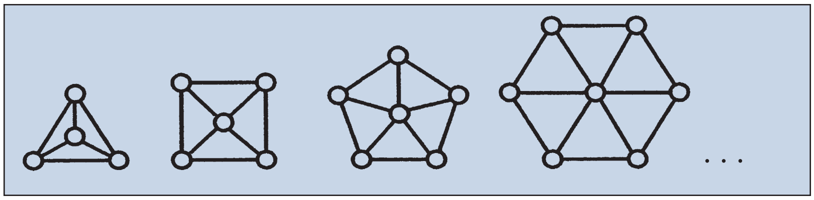


Fig. 5 The wheel family, k = 3, 4, 5, 6. A k-wheel is a k-cycle plus an extra vertex adjacent to all the others.

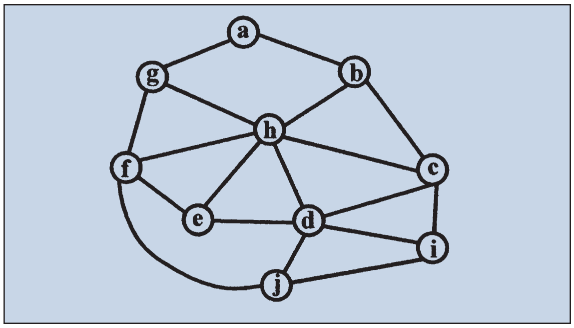


Fig. 6 Tricky graph for assessing understanding of coloring and proofs

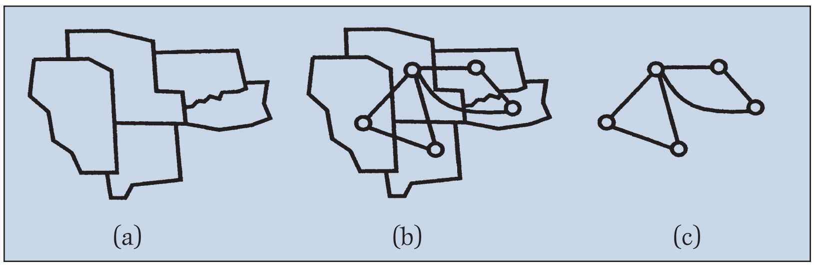


Fig. 7 Transforming map coloring into graph coloring. Given a planar map **(a),** assign a vertex to each region and join two vertices by an edge if the corresponding regions share a border **(b)**. A minimum coloring of the resulting graph **(c)** yields a minimum coloring of the map.

ing all possibilities. The Traveling Salesperson problem (TSP) is good for introducing this idea. In fact, the TSP is closely related to the important P versus NP problem, one of the seven problems for which the Clay Mathematics Institute has offered a million dollars for a solution (see COMAP 2003, p. 44, and Cipra 2002).

I start with a four-city example such as that given in *For All Practical Purposes* (COMAP 2003, pp. 34–38), shown in **figure 8.** The problem is to find the shortest route that starts and ends at a specific city and visits each other city exactly once. Later, I ask how selecting a different starting city affects the final answer.

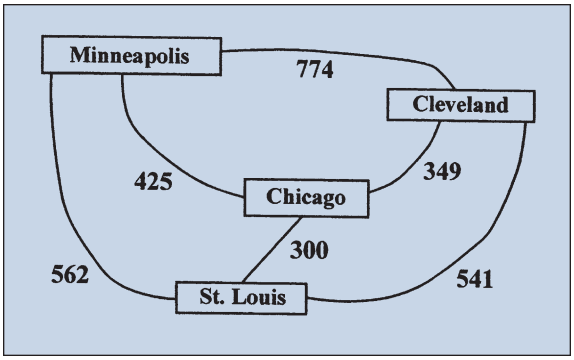


Fig. 8 Graph in which vertices represent four cities and the values on the edges are the distances in miles between cities. Source: Based on a problem from *For All Practical Purposes: Mathematical Literacy in Today's World,* 6/e by COMAP. ©2003 by W. H. Freeman Company. Used with permission.

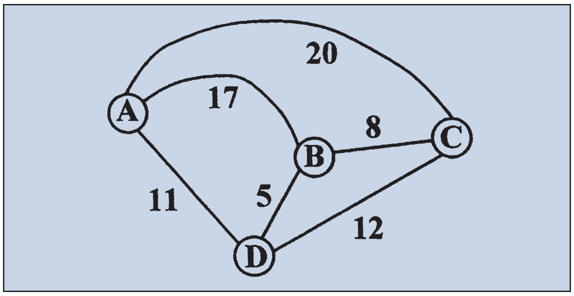


Fig. 9 Traveling Salesperson problem with four cities. The shortest route contains the longest edge—a counterexample to the conjecture that eliminating the longest edge always produces the shortest route.

Students are told to start at Cleveland. After trying several routes, students conclude that the shortest route is Cleveland—Chicago—Minneapolis—St. Louis—Cleveland (or the reverse route), which is 1877 miles. The following imaginary dialogue illustrates some of the ideas that usually arise.

Teacher: We now know that 1877 miles is possible, so it's an upper bound on the route length. How do we know that there is no shorter route?

Student: We tried eight other routes, and they were all longer.
T: How do we know that you didn't miss a route?
S: I know it's the shortest route because we eliminated the longest edge, from Minneapolis to Cleveland.
T: Now we're making progress. We have a conjecture to test (writing on board): "The shortest route cannot contain the longest edge." Let's check this on a different example [see **fig. 9**]. There are still four cities, A, B, C, and D, but I changed the distances. Find the shortest route starting and ending at C.
T: (after students try the example) Do you get the shortest route when you eliminate the longest edge?
S: No—the shortest route has length 44, but it uses the longest edge (A, C).
T: So eliminating the longest edge works in some cases, but it does not always work. So, unfortunately, we can't use this idea to prove that we have the shortest route.
S: I think you have to start with the shortest edge out of Cleveland (349 miles).
T: Do you always have to take the shortest edge out of the starting city?
S: Since you're going in a circle, it doesn't really matter where you start. If you started in Chicago, you wouldn't take the shortest edge.
S: So I guess that rule only works for Cleveland.
T: Let me summarize. We just had a new conjecture: "The shortest route always uses the shortest edge out of the starting city." However, you get the same shortest route starting in any of the four cities. If you start from Chicago instead, you don't use the shortest edge. So this is a counterexample that shows the conjecture is false.

At this point, the students have not yet proved that they have the shortest route, but they have generated ideas that turn out to be useful in practice. The strategies students have suggested for finding a shortest route, if applied repeatedly, are equivalent to standard greedy methods for getting an initial approximation to a good route when there are many cities (see COMAP 2003, pp. 41–45, for an introduction to this topic).

After students present their ideas, I return to the question "How do we know that you didn't miss a route?" I introduce two standard methods for systematic listing: (1) using an enumeration tree and (2) using lexicographic order, both illustrated in **figure 10.** (See also COMAP 2003, p. 37, which shows an enumeration tree for the example in **figure 8.**) Students usually realize that, in solving the TSP, they only have to check half the possible routes, since each route has an oppositely directed twin. Thus, with four cities, one needs to compare the lengths of only three routes.

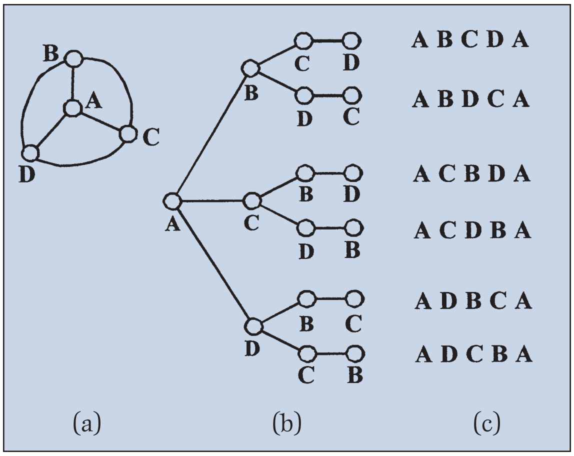


Fig. 10 **(a)** Four-vertex graph, where vertices *A, B, C, D* represent locations. **(b)** Enumeration tree for all routes starting and ending at *A,* oriented from left to right. Each path in the tree represents a route, listed to the right of the path. **(c)** Lexicographic ordering of routes, where $A < B < C < D$, reading from top to bottom. This is the same order as that generated by the tree, because we ordered the choices in each tree layer lexicographically.

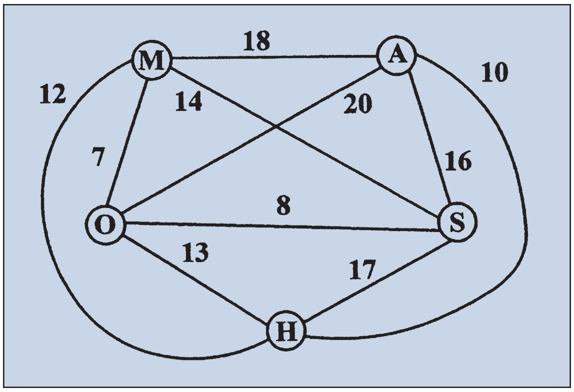


Fig. 11 The Hotel Shuttle problem. The vertices represent the airport (A) and four hotels (Omni, Hilton, Sheraton, and Marriott), and the values on the edges are the driving times between locations. What is a route that starts and ends at the airport, visits each hotel once, and has minimum time?

To check understanding, I might give a similar problem in which the starting city, but not the ending city, is specified. To help ensure that students go beyond simply memorizing the enumeration method for four cities, I have students generalize, using a five-location problem such as that in **figure 11.** Students are then usually ready to predict the number of routes in a six-city problem and to generalize to *n* cities. This counting exercise helps illuminate the limitations of proof by enumeration. In practice, TSPs with many cities are solved approximately or by using linear programming.

Another problem for both assessment and deepening of understanding is the Shortest Path problem. The goal is to find the shortest path between two given locations; visiting all other locations is not required (see Rosen 1999, chap. 7.6, or Cozzens and Porter 1987, sec. 3). Although an elegant solution method (Dijkstra's algorithm) exists, solving small problems by enumeration can help students understand systematic listing and motivate more efficient algorithms.

CONCLUSION

I have shown here how I use combinatorial optimization problems to introduce principles of mathematical reasoning. An important step is modeling the process of formulating conjectures and searching for counterexamples. From there one can introduce methods of proof, including direct implication, contradiction, use of previous results, and enumeration. Further optimization problems that could be used include the mail carrier problem (Chinese Postman problem; see Roberts 1997, pp. 113–15, or COMAP 2003, pp. 10–17); the Minimum Completion Time problem (COMAP 2003, pp. 52–56); the Minimum Spanning Tree problem (NCTM 2000, pp. 317–18, or COMAP 2003, pp. 45–49); and problems on scheduling or bin packing (COMAP 2003, pp. 76–98). I look for instances that are large enough to be interesting but small enough for students to be able to prove that a solution is correct. The richest problems are those with more than one solution or for which more than one method of proof can be used.

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