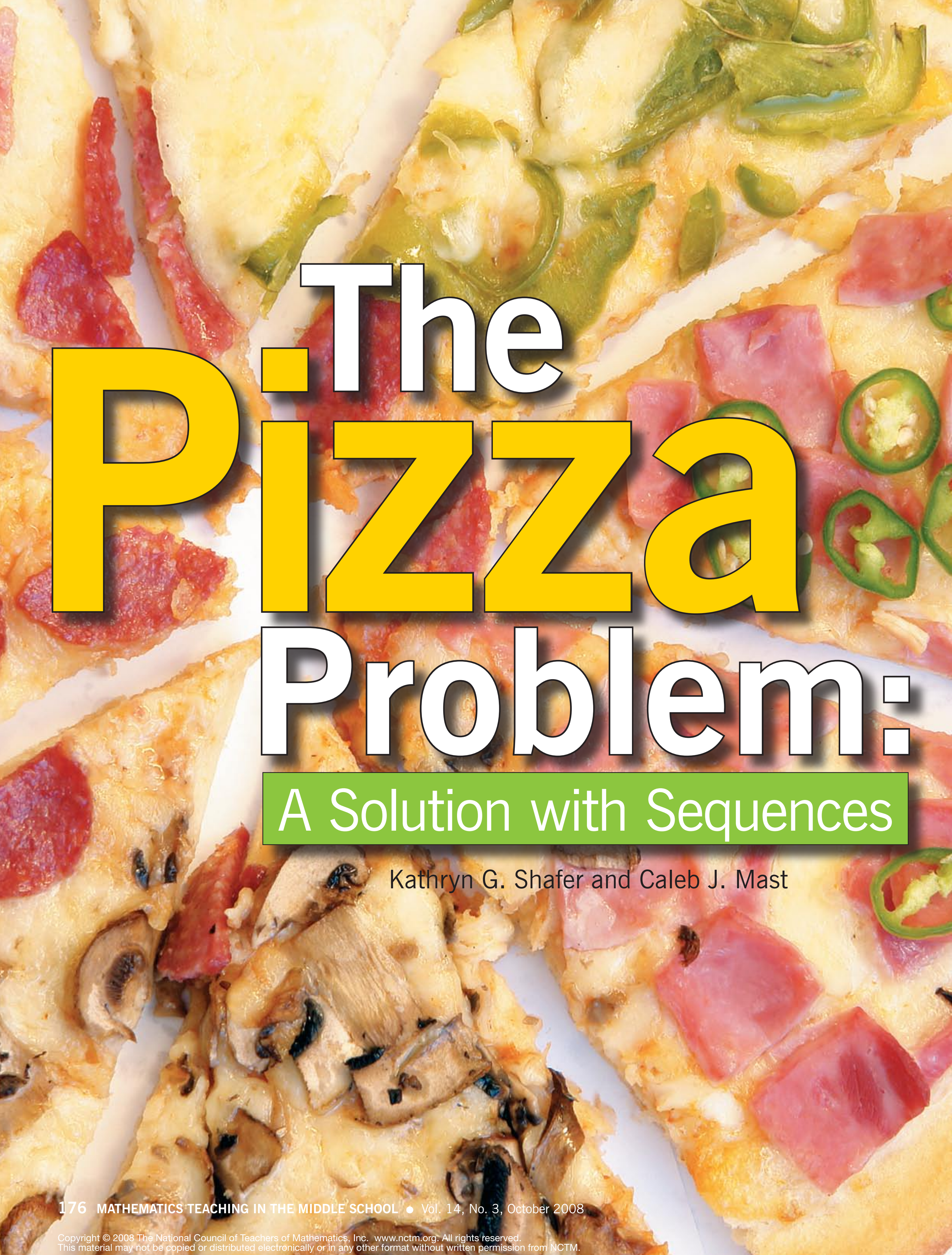

The Pizza Problem:

A Solution with Sequences

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As a mathematics educator working with preservice education majors, one of my primary goals is to provide students with problem-solving experiences. This is accomplished through the use of the "problem of the week," or POW. Each problem is selected with a different strategy in mind. The students work independently on one problem each week; suggestions and class discussions then occur when necessary. About a month into the fall semester, I assign the Pizza problem.

> A pizza restaurant has 10 different toppings for its cheese-and-tomato pizza: mushrooms, peppers, pepperoni, sausage, onion, anchovies, tuna, pickles, shredded wheat, and celery. How many different kinds of pizza can be made by varying the combination of toppings? (Hint: Try the same kind of problem with 1 topping, then 2 toppings, and so on, and look for a pattern.) (Sonnabend 2004, p. 56)

Many students struggle with this particular problem. Since the pizza can have up to ten toppings, they try to list and count each and every pizza possible. They discover that this leads to more pizzas than they would like. Eventually they begin to follow the suggested strategy, which in this case is to create an easier problem to solve, find a pattern, and then apply it to the original problem. For example, a common solution strategy used by students is to list 4 pizzas with up to two toppings (the no-topping cheese-and-tomato pizza, the mushroom pizza, the pepper pizza, and the mushroom and pepper pizza), then list the number of pizzas with up to three toppings, which is 8. To confirm the pattern, it is reasonable to list all 16 pizzas that can be made with up to four toppings. Since the students may be familiar with geometric sequences, they can apply the formula 2^n, where n is the number of toppings. So with ten toppings, the solution is 2^{10}, or 1024, pizzas. One semester, while going over an assignment, I was surprised by one student's work. Caleb had not followed the suggested strategy and instead completed the problem using a unique strategy that made sense to him. In the following text, Caleb will explain his procedure and thought processes. In "Reaching a Conclusion," I will highlight the mathematics that Caleb reinvented and reflect on the importance of keeping an open mind when hinting at solution strategies.

CALEB EXPLAINS THE PIZZA PROBLEM

Let's say that there are ten different toppings that one can use on a pizza. We want to show how many different combinations of toppings can be created using any number of the ten toppings. Start counting with the one combination when no toppings are used. This pizza is a cheese-and-tomato pizza. If only one topping is used to create a pizza, we can have only 10 pizzas because there are ten toppings to use. For two toppings, figure that there are ten choices for the first topping and then any of the remaining nine for the second topping. Multiplying these together produces the result of 45 different combinations of pizzas when you use two toppings.


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Let's stop here and start counting how many combinations of pizzas can be created if we start counting down from the highest number of toppings we can use. I wanted to start at the highest number because I suspected that the total number of combinations for nine toppings and ten toppings would be relatively small and easy to count. If we use ten toppings, we can only create one pizza. If we use nine toppings, there are ten different combinations of toppings for a pizza. (Think about the ten different ways to leave off just one topping.) If we count how many pizzas can be made with a combination of eight toppings, we find that the number is 45. (Consider all the different possible combinations of leaving off two toppings, just as we previously thought about using two toppings.) As shown in **figure 1,** starting on each end of my table, the number of pizzas is the same and grows as they reach the center. This growth's symmetry resembles a palindrome. At this point, the problem becomes more difficult. We have to count how many different combinations of pizzas can be created with three toppings. Instead of thinking in terms of pizza toppings, think in terms of letters. For example, let mushrooms be represented by the letter *a*, and let peppers be represented by the letter *b*. We must first start with all the mushroom combinations, which are the combinations starting with *a-b-c*, *a-b-d* up to *a-i-j* (see **fig. 2**). To count the first set, shown in **figure 2**, we notice the last letter starts at *b* and ends at *j*, giving us eight different combinations. In **figure 2**, the last letter in the set will only range from *c* to *j*, giving us a combination of seven different topping mixtures. As we can observe, a pattern is beginning to form, which can be followed all the way to the single *a-i-j* combination. This gives us 36 different three-topping pizzas with mushrooms, which was found by counting all the rows in **figure 2**. The totals for all of the different three-topping pizza combinations can be found by repeating this process.

There are 36 pizzas that have mushrooms as the first of its three toppings, as counted in **figure 2**. Next, if we count all of the pizzas that have peppers as the first of its three toppings, disregarding all the mushroom pizzas we have already counted, we find that there are 28. **Figure 3** organizes all the three-topping pizzas by counting the combinations by *first* topping and disregarding all toppings already counted as *first*. This counting is shown in **figure 3**. In addition, the level 1 differences are shown for the pizzas counted at each stage. Notice that the differences decrease by 1 across the table.

The concept that the level 1 differences decrease by 1 as the sequence continues, as shown in **figure 3**, is important. This idea is used later in the problem. The number of pizza combinations for three toppings is 120, found by adding the number of pizzas across the second row. Since three-topping pizzas have 120 combinations, then seven-topping pizzas will also have 120 combinations, as seen in the symmetry in **figure 1**.

Once this pattern was found, I computed all the pizza combinations using four toppings. Using the same counting method as I used in **figure 2** and **figure 3**, I found that the total number of combinations equaled 210; therefore, six-topping pizzas will also have 210 different combinations. Once all the combinations have been found for the four-topping pizzas, we can begin to analyze the results. I decided to look at the differences again (see **fig. 4**) and found that the difference sequence has a pattern. As we see in **figure 4**, the difference of the difference (level 2 differences) decreases by 1 as you move across the fourth row of the table.

This pattern frustrated me until I realized that these are the same numbers I had in the combinations of three toppings; note the repetition of the 28, 21, 15, 10, 6, 3 sequence. Then I noticed that the level 2 differences with the four-topping pizzas is the same as the level 1 differences with the three-topping pizzas.

Fig. 1 Total number of pizzas for each number of toppings: Incomplete

Number of toppings	0	1	2	3	4	5	6	7	8	9	10
Number of pizzas	1	10	45	?	?	?	?	?	45	10	1

Fig. 2 Number of three topping pizzas with mushrooms

a	*b*	*c*
a	*b*	*d*
a	*b*	*e*
a	*b*	*f*
a	*b*	*g*
a	*b*	*h*
a	*b*	*i*
a	*b*	*j*

a	*c*	*d*
a	*c*	*e*
a	*c*	*f*
a	*c*	*g*
a	*c*	*h*
a	*c*	*i*
a	*c*	*j*

a	*d*	*e*
a	*d*	*f*
a	*d*	*g*
a	*d*	*h*
a	*d*	*i*
a	*d*	*j*

a	*e*	*f*
a	*e*	*g*
a	*e*	*h*
a	*e*	*i*
a	*e*	*j*

a	*f*	*g*
a	*f*	*h*
a	*f*	*i*
a	*f*	*j*

a	*g*	*h*
a	*g*	*i*
a	*g*	*j*

a	*h*	*i*
a	*h*	*j*

a	*i*	*j*

REACHING A CONCLUSION

Seeing this pattern helps find a way to get the number of combinations of five-topping pizzas without counting them all, which would be time-consuming. I found that a relationship exists between pairs of combinations and their differences, as shown in **figure 3**'s diagram for three-topping pizzas:

Number of pizzas: 36 28 21

Level 1 differences: 8 7

The first two adjacent combinations form the "triangular" equation 28 + 8 = 36 and the next two form the equation 21 + 7 = 28. In **figure 4**'s four-topping pizza diagram, level 2 differences demonstrate which two "triangular" equations are needed to reach 84. Specifically, we use 21 + 7 = 28 and 56 + 28 = 84. If this pattern holds, we will have three "triangular" equations, using common differences to find the different number of five-topping pizzas.

To begin, count the easiest combinations of five toppings by counting backward. Count 5 letters back from *j*, and you will get *f*. Therefore, your last combination will be *f-g-h-i-j*. Then count how many combinations there will be when you start with the previous letter *e* by using the method described earlier. There are 5 such combinations (*e-f-g-h-i*, *e-f-g-h-j*, *e-f-g-i-j*, *e-f-h-i-j*, and *e-g-h-i-j*). Then count from *d*, and you will find 15 different combinations. Finally, count

Fig. 3 Three-topping pizzas

Constant topping	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*
Number of pizzas	36	28	21	15	10	6	3	1	*	*

Level 1 differences 8 7 6 5 4 3 2

* There are no three-topping pizzas to consider.

Fig. 4 Four-topping pizzas

Constant topping	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*
Number of pizzas	84	56	35	20	10	4	1	*	*	*

Level 1 differences 28 21 15 10 6 3

Level 2 differences 7 6 5 4 3

* There are no four-topping pizzas to consider.

Fig. 5 Five-topping pizzas: Incomplete

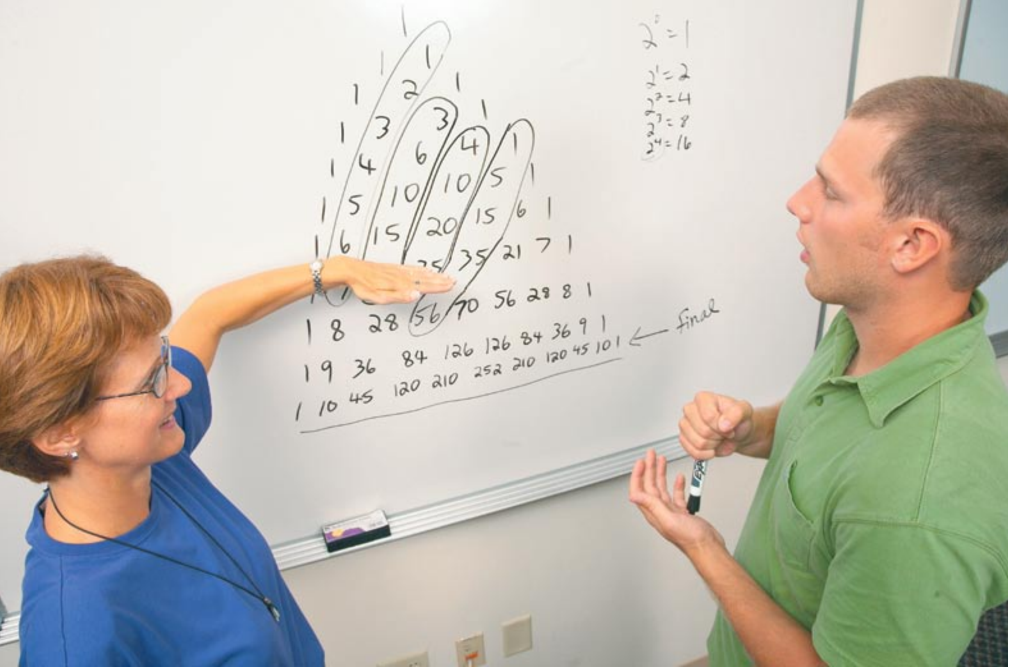
Constant topping	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*
Number of pizzas			35	15	5	1	*	*	*	*

Level 1 differences $\boxed{}$ $\boxed{}$ $\boxed{20}$ $\boxed{10}$ $\boxed{4}$

Level 2 differences $\boxed{}$ $\boxed{}$ $\boxed{10}$ $\boxed{6}$

Level 3 differences $\boxed{6}$ $\boxed{5}$ $\boxed{4}$

* There are no five-topping pizzas to consider.



from *c*, which results in 35 different combinations. Using the pattern of increasing by 1 to generate the level 3 differences, we have enough numbers to create the table for the five-topping pizzas (see **fig. 5**).

EXPLORING CALEB'S WORK

This part of the problem motivated Caleb to produce a creative strategy. He had counted the 35 pizzas for the pepperoni topping (second column) and did not want to list and count the pepper and mushroom toppings for the five-topping pizzas. It would be too time-consuming, and the chance for error would be high. The "triangular" equation pattern explained earlier can be used here to calculate the missing numbers for the five-topping pizzas (see **fig. 6).** To find the number of combinations of six toppings, the number-of-pizzas row in **figure 6** is added and we find 252 combinations for five-topping pizzas. **Figure 7** shows all the numbers for the different combinations of pizzas. Now that we know the number of combinations with exactly one to ten toppings, we add these numbers and find that there are 1024 ways to prepare a pizza with a choice of ten toppings.

After the problem was finished, we discussed Pascal's triangle (**fig. 8**) and how it might tie into this work. In the tenth row of the triangle, the numbers are the same as the number of pizzas. For example, the second number in the tenth row of Pascal's triangle is 10 and the third is 45, which is the same as the number of pizzas with one topping and the number of pizzas with two toppings. The second diagonal on the triangle contains the level 3 differences, the third diagonal contains the level 2 differences, the fourth diagonal contains the level 1 differences, and the fifth diagonal contains the number of pizzas with five toppings.

REFLECTIONS ON CALEB'S WORK

Caleb suspected that symmetry was at work in the problem when the number of pizzas with zero and ten toppings was the same and the number for one and nine toppings was the same. He confirmed that the symmetry continued for the number of pizzas with two and eight toppings by counting them. This reduced the problem to finding the number of pizzas for three, four, and five toppings to complete the table. Caleb then began working on the number of three-topping pizzas, given ten toppings. Not realizing that this is ${}_{10}C_3$, he wrote them all out.

I was fascinated by the next part of Caleb's strategy. When he originally completed this problem, his strategy was to list and count all the possible pizzas that could be produced with up to ten toppings. Although this was time-consuming, his paper contained these lists as well as the number patterns explored above. What really pushed him to examine the patterns was the listing and counting involved in finding all the five-topping pizzas.

Fig. 6 Five-topping pizzas: Complete

Constant topping	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*
Number of pizzas	126	70	35	15	5	1	*	*	*	*

Level 1 differences 56 35 20 10 4

Level 2 differences 21 15 10 6

Level 3 differences 6 5 4

* There are no five-topping pizzas to consider.

Fig. 7 Total number of pizzas for each number of toppings: Complete

Number of toppings	0	1	2	3	4	5	6	7	8	9	10
Number of pizzas	1	10	45	120	210	252	210	120	45	10	1

Fig. 8 Pascal's triangle

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

When Caleb counted out the 35 pepperoni pizzas and knew that he had two more sets to go, he stopped and decided to exploit the patterns he had discovered to find the solution for ${}_{10}C_5$.

Since the presentation of a strategy is an element I strive for as an educator, Caleb's initial paper was problematic. The page was covered with his work, which ran in many directions. Because of a lack of organization, he had made one counting error that caused the final solution to be too large. I knew that he had produced some valid mathematics but did not know what caused the error. When he explained his thinking, I was astounded by the quality of the mathematics he had completed. We found the computation error and confirmed that his strategy worked.

LESSONS LEARNED

As an educator, I learned two valuable lessons. First, the presentation that appears to be a mess may be worth a second look, and an interview can help uncover the student's thought process. Second, it may not be a good idea to suggest the standard strategy that should be used, since in this case it would have stifled a meaningful adventure in mathematics for Caleb. In fact, while sharing Caleb's solution strategy with colleagues, a mathematician asked if any of the students had thought of an approach using combinatorics. The mathematician described the process of creating a pizza at a local restaurant by starting at the beginning of a line where a server would offer one of ten toppings. While walking past the topping choices, you would answer either yes or no. Since you only have two choices for each of the ten responses, the solution is 2^{10}, or 1024, possible pizzas. This illustration lends further support to the caution of prompting students with a predetermined, specific solution strategy.

REFERENCE

Sonnabend, Thomas. *Mathematics for Teachers: An Interactive Approach for Grades K–8*. Belmont, CA: Brooks/Cole—Thompson Learning, 2004.