Roger Ford

Discovering and Exploring Mandelbrot Set Points with a Graphing Calculator

The Mandelbrot set is one of the most beautiful and complicated objects in all of mathematics. The border of the set is infinite; no matter how many times you magnify the set, dazzling new images appear. Mandelbrot's discovery of the set and his subsequent work on fractals and recursive functions would not have been possible without the aid of the computer. Although the mathematics behind the Mandelbrot set is simple, creating the Mandelbrot set images involves millions of calculations. Mandelbrot's genius allowed him to see in these wild images a tool that helps explain many complex processes and problems in mathematics and science. Because of Mandelbrot's work, advances have been made in medicine, chaos theory, digital picture compression, and in modeling natural processes.

Technology has empowered students and mathematicians to solve problems and understand them in new ways. Using the graphing calculator to discover and explore the Mandelbrot set allows students to see the mathematics behind a great mathematical discovery and to see how valuable technology can be in solving and understanding problems.

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BACKGROUND

The Mandelbrot set is a set that consists of some of the points in the complex-number plane (the Argand plane). The points in the Mandelbrot set are determined by using a recursive formula.

The complex-number plane and the Cartesian plane are very much alike. In the Cartesian plane, two real numbers (x, y) determine a point. In the complex-number plane, a complex number consisting of a real number and an imaginary number $(x + yi)$ determine a point. See **figure 1.**

Mandelbrot's recursive formula for finding all the points in the complex-number plane that are a

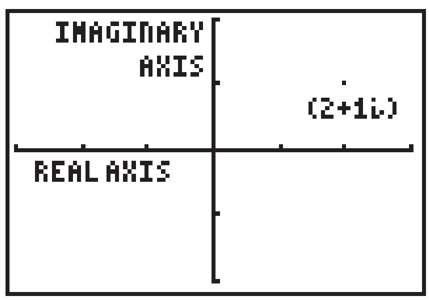

Fig. 1 Complex-number plane with point $(2 + 1i)$

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part of the Mandelbrot set starts by testing the point Z_0. This point's coordinates are squared, and the coordinates of the original point are added to create a new point Z_1. For example, let our original point Z_0 be $-0.4 + 0.3i$. Then

$$\begin{aligned} Z_1 &= (Z_0)^2 + Z_0, \\ Z_1 &= (-0.4 + 0.3i)^2 + -0.4 + 0.3i \\ &= 0.07 - 0.24i + -0.4 + 0.3i \\ &= -0.33 + 0.06i. \end{aligned}$$

From this point forward, the formula for finding the next new point or iteration point (Z_n) follows the general formula $Z_n = (Z_{n-1})^2 + Z_0$. For example, the original point (Z_0) and iteration points Z_2 and Z_3 are computed as follows:

$$\begin{aligned} Z_2 &= (Z_1)^2 + Z_0 \\ &= (-0.33 + 0.06i)^2 + -0.4 + 0.3i \\ &= -0.2947 + 0.2604i; \end{aligned}$$

$$\begin{aligned} Z_3 &= (Z_2)^2 + Z_0 \\ &= (-0.2947 + 0.2604i)^2 + -0.4 + 0.3i \\ &= -0.38096007 + 0.14652024i \end{aligned}$$

The calculations become cumbersome very quickly. **Figure 2** shows the graph of the original point, Z_0, and the next three iteration points derived by using Mandelbrot's recursive formula. For each of these graphs, the coordinates of the original point (Z_0) appear at the top of the graphing calculator screen, and the original point is plotted as a rectangle. All the subsequent iteration points are shown as dots. The distance between the tick marks on both the real and imaginary axes represents 0.1 units.

As this recursive process continues, one of three patterns emerges for the iteration points. Sometimes these points tend toward one location that is very close to the origin (0, 0). Sometimes they move farther and farther away from the origin, and sometimes they bounce around but are never far from the origin. The original points that generate the iteration points that never get very far from the origin or that tend toward one location are the points in the Mandelbrot set. The previous example point $Z_0 = -0.4 + 0.3i$ generates iteration points that tend toward the location $-0.3264 + 0.1815i$, rounded to four decimal places. (See **fig. 3.**)

The program MBPOINTS, **program 1,** on page 44, was designed for use with a TI-83 calculator and was used to create the graphs in **figure 3, figure 5,** activity sheet problem 7, and the solutions to problem 7. Interesting graphs of other Z_0 points can be generated by using this program.

For this Z_0, the points Z_{50}, Z_{100}, and Z_{1000} are all very close to one another; and Z_0 is a point in the Mandelbrot set. To test whether an original point

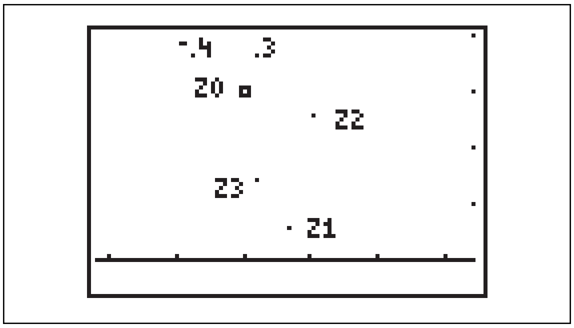

Fig. 2 First three iteration points of $Z_0 = -0.4 + 0.3i$

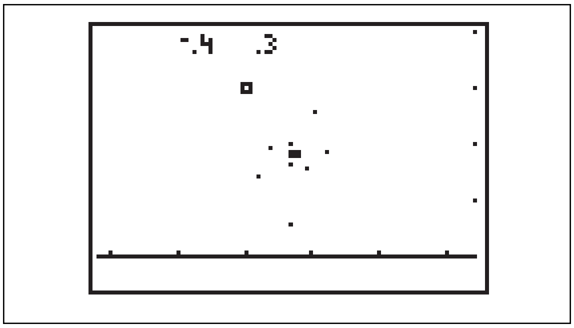

Fig. 3 Fifty iteration points of $Z_0 = -0.4 + 0.3i$

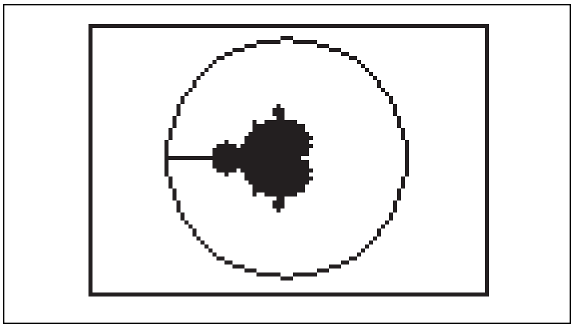

Fig. 4 Mandelbrot set with circle of radius 2

will generate iteration points that continue to get farther and farther away from the origin, calculating the Euclidean distance from the origin to the point Z_n is necessary. It has been proved that if the distance between the origin and Z_n ever exceeds two units from the origin, then subsequent iteration points will continue to move farther away from the origin. **Figure 4** shows the Mandelbrot set and a circle with a radius of 2 units from the origin. If any iteration point Z_n has coordinates outside the circle, the original Z_0 is not a part of the Mandelbrot set.

For example, let the original point be $Z_0 = -0.9 + 0.6i$. Then the distance of Z_0 from the origin is $\sqrt{0.9^2 + 0.6^2}$, or approximately 1.0817. Since the distance from the origin to the original point is less than or equal to 2 units, Z_0 may still be a part of the Mandelbrot set. However, as the recursive process is continued for Z_0, the distance of the iteration point from the origin eventually increases beyond 2 units, as shown in **table 1.** Since the Z_4 distance is greater than 2 units from the origin,

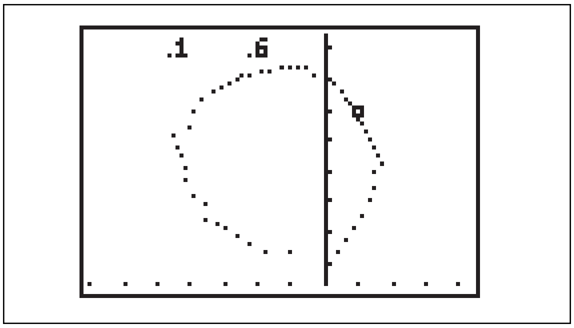

Fig. 5 Iteration points of $Z_0 = 0.1 + 0.6i$

the iteration points will stray even farther away; and the original point Z_0, $(-0.9 + 0.6i)$, is not part of the Mandelbrot set.

In **figure 5,** the iteration points generated from the original point $Z_0 = 0.1 + 0.6i$ stay close to the origin but never settle into a pattern. The plot of all iteration points of any Z_0 is called the *orbit* of Z_0.

Table 2 shows that for $Z_0 = 0.1 + 0.6i$, the iteration points from Z_{46} to Z_{50} stay within a distance of 2 units from the origin.

This original point $Z_0(0.1 + 0.6i)$ is a part of the Mandelbrot set because each iteration point stays within a radius of 2 units from the origin.

TABLE 1

Distance from the Origin

Z_0	1.0817 (rounded to 4 decimals)	
Z_1	0.6580	
Z_2	1.3878	
Z_3	1.7171	
Z_4	3.7774	$Z_4 = -1.4108 + 3.50400i$

TABLE 2

Iteration Points from Z_{46} to Z_{50} for $Z_0 = 0.1 + 0.6i$

	Real	Imaginary	Distance from Origin
Z_0	0.1	$0.6i$	0.6083
Z_{46}	0.09379	$0.57347i$	0.5811
Z_{47}	−0.2201	$0.70758i$	0.7410
Z_{48}	−0.3522	$0.28857i$	0.4553
Z_{49}	0.1408	$0.39671i$	0.4210
Z_{50}	−0.0376	$0.71171i$	0.7127

TEACHER'S GUIDE

This activity is designed for students in algebra 2 and higher levels of mathematics. Students must know how to add complex numbers, multiply complex numbers, graph in the complex-number plane, and calculate using a recursive formula. Each student needs a graphing calculator with **program 2,** the MANDEL program. This article includes instructions for using the TI-83 or TI-83 Plus with the MANDEL program. If the students are using a different graphing calculator, the teacher will need to modify the program.

The objective of the activity is for the students to become aware of the mathematics behind the Mandelbrot set and the power of technology in mathematical processes.

The activity consists of two parts: a cooperative learning activity and the activity sheets. In the cooperative learning activity, the students, as a team, discover points in the complex-number plane that are in the Mandelbrot set. My classes spend parts of three days doing this activity. On the first day, I introduce the Mandelbrot set and explain how it is derived. On the second day, the students enter **program 2** into their calculators and find the iteration value for various points in the complex-number plane. They record these values on individual grids. The iteration values determine which original points are a part of the Mandlebrot set and how quickly some iteration points leave the radius of 2 units from the origin. For example, the iteration value of $Z_0(-.25 + 1.10i)$ is 4 because the fourth iteration generates an iteration point with a distance from the origin that is greater than 2 units. An initial point, Z_0, is a member of the Mandelbrot set if it has an iteration value of 51, meaning that none of the iteration points generated by Z_0 have gone beyond a circle of radius 2 centered at the origin.

Several days after completing that part of the activity, the students receive a photocopy of the master complex-number plane grid with all the point iteration values. The students then derive an iteration-value coloring scale and color their photocopy. Teachers can use graph paper to create the complex-number-plane grid.

MANDEL program

The MANDEL program, **program 2,** on page 44, is designed to allow students to enter the real coordinate number and the imaginary coordinate number of a point in the complex-number plane. For example, suppose that a student wants to enter the complex point $0.123 + -0.567i$. After the R=? prompt, the student should type .123; and after the I=? prompt, the student should type –.567. When the student presses ENTER, the screen shown in **figure 6** should appear.

Iteration 1 means that after using the recursive formula one time, point $Z_1 = -0.18336 - 0.706482i$, and it is 0.7298888312 units from the origin. Pressing ENTER one more time produces the screen shown in **figure 7.**

Iteration 2 means that after using the recursive formula for a second time, point $Z_2 = -0.3424959267 - 0.307918921i$, and it is 0.4605621801 units from the

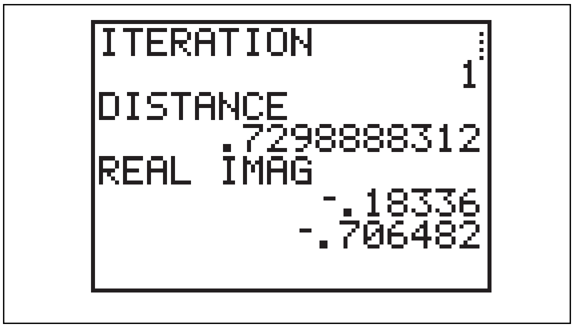

Fig. 6 First iteration of Z_0 = 0.123 + –0.567i

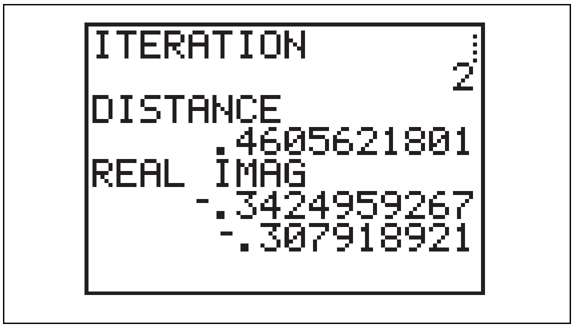

Fig. 7 Second iteration of Z_0 = 0.123 + –0.567i

origin. When the user continues to press the **ENTER** key, the screen in **figure 8** eventually appears.

When iteration 51 appears and the word Done appears, the program has tested the point Z_0 for fifty iterations and found that none of the points created is more than 2 units from the origin. Therefore, Z_0 (0.123 + –0.567i) is a point in the Mandelbrot set. Had the program not reached fifty-one iterations, the point would not be considered part of the Mandelbrot set. The decision to test fifty iterations to get the number 51 was made arbitrarily. By using fewer iterations, the Mandelbrot-set image is less refined. A larger iteration value yields a slightly more accurate picture, but it also slows the process. In this day of instant computer feedback, students often believe that fifty iterations is "so slow."

Figure 9 shows the final screen of Z_0 = –0.7 + 0.7i. After four iterations, the distance from the origin exceeds 2 units, so Z_0 is not a part of the Mandelbrot set.

Once the students type in the MANDEL program, they should test the program to make sure that it gives the same answers as some of the previous examples. A single wrong sign or a parenthesis in the wrong place results in incorrect answers.

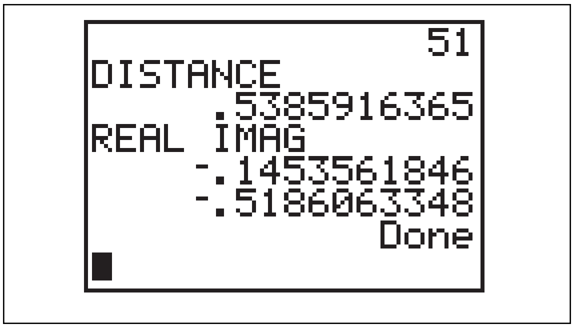

Fig. 8 Fifty-first iteration of Z_0 = 0.123 + –0.567i

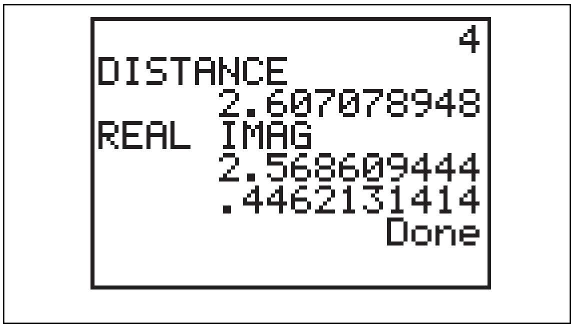

Fig. 9 Fourth iteration of Z_0 = –0.7 + 0.7i

Cooperative activity

Students use **program 2** to fill in each cell of the complex-number-plane grid with an iteration value. The iteration value comes from entering a complex-number-plane point in the MANDEL program. The cells on the complex-number-plane grid range from –2 to 0.8 on the real axis and from –1.2i to 1.2i on the imaginary axis. Each axis is divided into increments of 0.05. **Figure 10** shows a corner of the grid. The total number of cells is 2793, so the task of finding an iteration value for each cell can be tedious.

Students are usually given the assignment of finding all the iteration values in one row or column. For a column, a student might find all the iteration values when the real part is –1.5 and the imaginary part varies from –1.2i to 1.2i. Since the Mandelbrot set is symmetrical with respect to the real axis, each student's work can be cut almost in

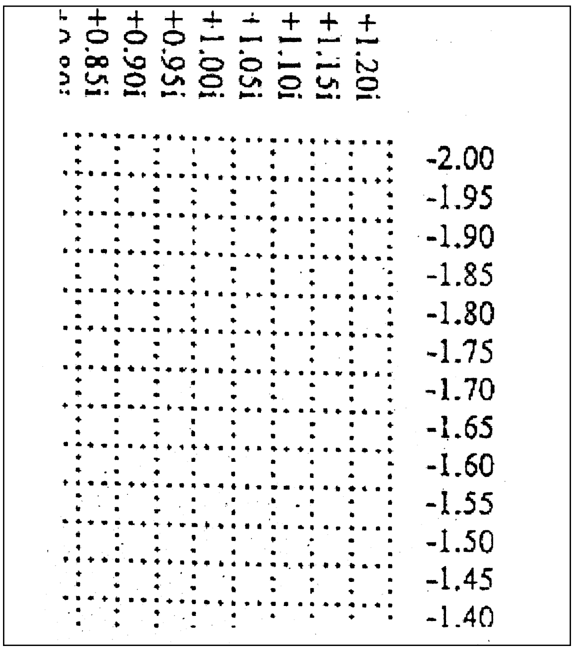

Fig. 10 A corner of the grid

TABLE 3

Iteration Values for Grid Cells

		Real					
		−0.25	−0.20	−0.15	−0.10	−0.05	0
Imaginary	1.10i	4	18	7	4	4	3
	1.05i	5	7	11	5	5	4
	1.00i	5	6	10	5	10	51
	0.95i	6	7	9	22	9	6

Iteration values	Color
1	blue
2	green
3–4	red
5–9	orange
10–50	yellow
51	black

Fig 11 Coloring scale

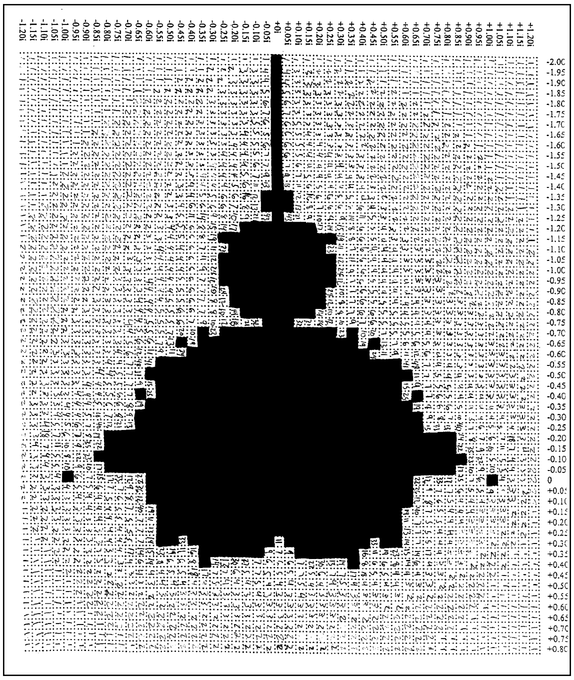

Fig. 12 A student's graph

half by assigning the student a real number and the nonnegative imaginary numbers. Also, the students will find that their work goes much faster if they eliminate the PAUSE statement in the second-to-last line of the program. After the students have found all the iteration values for their assigned cells, those values are entered on a master complex-number-plane cell grid. **Table 3** is an example that shows part of the grid.

This cooperative activity then turns back into an individual activity. Each student receives a photocopy of the completed grid. Only cells with the value of 51 in them represent points in the Mandelbrot set. Cells with values that are less than 51 are not part of the Mandelbrot set. The smaller the value in the cell, the quicker Z_0 was eliminated from possibly being a member of the Mandelbrot set. By tradition, all points in the Mandelbrot set are colored black because Benoit Mandelbrot first saw the set in that way. For the other cells, the students should devise a coloring scale, similar to the one in **figure 11,** which is based on the iteration values. The ranges of the iteration values do not have to be equal. When this scale is completed, the students can color their graphs. **Figure 12** shows a black-and-white version of a student's graph.

As the final part of the Mandelbrot set project, my students used the MANDEL program to find the iteration values for a small portion of the Mandelbrot set. The more than 29,000 one-inch squares were painted a color on the basis of their iteration values. Although I thought that the project would take about three months to complete, it took almost the entire school year. The issue cover shows the completed mural.

The activity sheets

During the second part of this activity, the students complete activity sheets that are designed to reinforce and stimulate students' thinking on the processes that create the set, on the parameters that affect the set, and on the traits that are exhibited by the set's points. Completing the activity sheets takes students more than one regular class period because of the graphs in problem 7. The teacher may want to help the students find the graphing window for the three graphs in problem 7.

CONCLUSION

Gary Kasparov is the first world chess champion to lose a chess match to a computer. When he was done with the match, he said that he would like to team up with the computer and together they could produce beautiful chess. Many great discoveries and advances in mathematics and science will occur because of the joint efforts of humans and computers working together. Computers and calculators are more than databases, game machines, and word processors. Making students comfortable with technology and helping them become aware of the power of technology helps prepare them for their future.

SOLUTIONS

1. *a)* $Z_4 = -0.55 + 0.72i;$
   distance from the origin = 0.906

   b) $Z_5 = 1$; distance from the origin = 1
2. Mandelbrot's formula is recursive because find-

ing the next answer in a sequence, Z_n, requires using the previous sequence answer, Z_{n-1}.

3. Since the point –0.55 + 0.65i takes forty-eight iterations before it escapes beyond a radius of 2 units, the lowered iteration standard allows the point to be included as a point in the Mandelbrot set. The standards for membership in the Mandelbrot set were lowered to forty iterations so more points are included in the set and the set becomes slightly larger. It thus tends to be a less accurate representation of the set.
4. Since the point –0.75 + 0.05i escapes the radius of 2 after sixty-two iterations, the heightened iteration standard forces the point out of the Mandelbrot set. With the increased iteration standard, some of the points that were formerly in the set are no longer in the set and the set is slightly smaller. It thus tends to be a more accurate representation of the set.
5. The regions on the graph of the Mandelbrot set graph that are not colored black represent the points that are not in the Mandlebrot set. The colors represent different iteration values at which the iteration points are outside a radius of 2 units from the origin.
6. *a*) The iteration points from 45 to 50 become fixed at one point.

   b) The iteration points from 45 to 50 oscillate between two fixed points, or attractors, giving this Z_0 a period of 2.

   c) The iteration points from 45 to 50 oscillate between three fixed points, or attractors, giving this Z_0 a period of 3.
7. *a*) The iteration points appear to be spiraling inward to some fixed point near 0.25 + 0.4i. See **figure 13.**

   b) The iteration points form an oval but do not seem to be oscillating or tending toward a fixed point. See **figure 14.**

   c) The iteration points have come from two directions and are tending toward a fixed point near –0.46 + 0.05i. See **figure 15.**
8. The Mandelbrot set was not discovered until the 1970s because the power of computers and calculators to perform thousands of mathematical operations was necessary to obtain a good idea of what the set would look like. The answers to the most interesting features of the Mandelbrot set will vary from student to student depending on how much the students know about the set. Some possible answers include the infinite border of the set, the unusual shape of the set, the orbits of the points in the set, the color pattern for the points surrounding the set, the totally connected nature of the set, and that the set's shape changes because of altering the set inclusion iteration values.

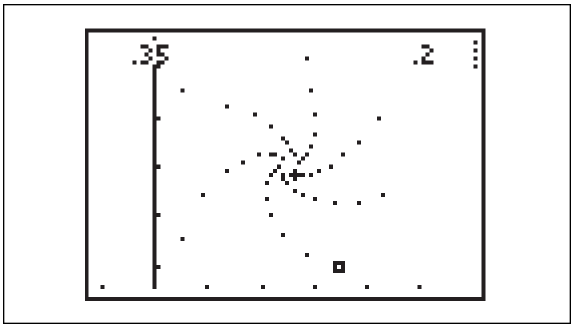

Fig. 13 Iteration points of Z_0 = 0.35 + 0.2i

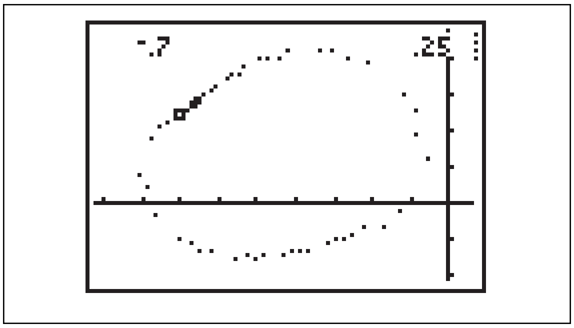

Fig. 14 Iteration points of Z_0 = –0.7 + 0.25i

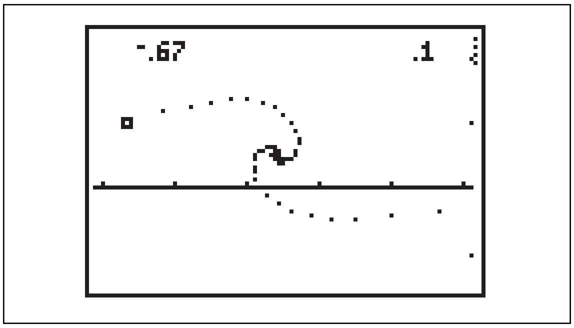

Fig. 15 Iteration points of Z_0 = –0.67 + 0.1i

REFERENCE

Devaney, Robert L. *An Introduction to Chaotic Dynamical Systems.* Redwood City, Calif.: Addison-Wesley Publishing Co., 1989.

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(Programs 1 and 2 appear om page 44; Worksheets appear on pages 45–46.)

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PROGRAM 1: MBPOINTS

```
Prompt R
Prompt I
R→Xmin
R→Xmax
I→Ymin
I→Ymax
.1→Xscl
.1→Yscl
R→S
I→J
0→C
0→D
ClrHome
Disp "CALCULATING"
Repeat D>2 or C>50
R→L1(C+1)
I→L2(C+1)
C+1→C
R²–I²+S→A
2*R*I+J→B
√(A²+B²)→D
A→R
B→I
If R<Xmin
Then
R→Xmin
End
If R>Xmax
Then
R→Xmax
End
If I<Ymin
Then
I→Ymin
End
If I>Ymax
Then
I→Ymax
End
Output(1,13,C)
End
Xmax-Xmin→M
Ymax-Ymin→N
If .6667*M>N
Then
Xmin-.1*M→Xmin
Xmax+.1*M→Xmax
(Ymin+Ymax)/2→L
(.6667*M+.2*M)/2→K
L-K→Ymin
L+K→Ymax
Else
Ymin-.1*N→Ymin
Ymax+.1*N→Ymax
(Xmin+Xmax)/2→L
(1.5*N+.2*N)/2→K
L-K→Xmin
L+K→Xmax
End
Disp "GRAPH READY"
GridOff
AxesOn
DispGraph
Pt-On(L1(1),L2(1),2)
For(Z,1,C-1)
Pt-On(L1(Z+1),L2(Z+1),1)
End
Text(1,10,S)
Text(1,80,J)
Pt-On(L1(51),L2(51),3)
Pause
```

PROGRAM 2: MANDEL

```
Prompt R
Prompt I
R→S
I→J
0→D
0→C
Repeat D>2 or C>50
ClrHome
C+1→C
R²-I²+S→A
2*R*I+J→B
A→R
B→I
R→L1(C)
I→L2(C)
Disp "ITERATION"
Disp C
Disp "DISTANCE"
√(A²+B²) →D
Disp D
Disp "REAL IMAG"
Disp A
Disp B
Pause
End
```

Discovering and Exploring Mandelbrot Sets

Sheet 1A

1. Use Mandelbrot's recursive formula, $Z_n = (Z_{n-1})^2 + Z_0$, to find the coordinates of the point following each given point and its distance from the origin.
   a) $Z_3 = 0.2 + 0.3i$; $Z_0 = -0.5 + 0.6i$
   b) $Z_4 = -1 + 0.5i$; $Z_0 = 0.25 + 1i$

2. Explain why Mandelbrot's formula $Z_n = (Z_{n-1})^2 + Z_0$ is a recursive formula.

3. Suppose that the maximum number of iterations in the MANDEL program is reduced from 50 to 40. How does this adjustment affect the point $Z_0 = -0.55 + 0.65i$? How does the appearance of the Mandelbrot set change?

4. Suppose that the maximum number of iterations in the MANDEL program is increased from 50 to 100. (Hint: In the program, change the seventh line from $C > 50$ to $C > 100$.) How does this adjustment affect the point $Z_0 = -0.75 + 0.05i$? How does the appearance of the Mandelbrot set change?

5. Explain what the regions on the graph of the Mandelbrot set that are not colored black represent.

For the next two problems, add the following two lines on the fourteenth and fifteenth lines of the MANDEL program:

```
R→L1(C )
I→L2(C )
```

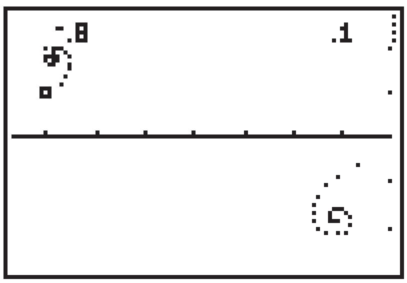
These lines put the real and imaginary values for the fifty iteration points in lists L1 and L2, respectively.

6. The Mandelbrot set consists of circular regions, called *bulbs,* that are surrounded by an infinitely complex border. For the given Mandelbrot set bulbs, select a Z_0 and explain the pattern that you find for iteration points numbered 45 through 50.
   a) A point from the middle of the largest bulb of the Mandelbrot set that does not have a 0 coordinate

   b) A point from the middle of the second-largest bulb of the Mandelbrot set that does not have a 0 coordinate

   c) A point from the middle of the third-largest bulb of the Mandelbrot set

7. The ordered pattern (orbits) of the iteration points in the Mandelbrot set can be interesting. For $Z_0 = -0.8 + 0.1i$, the graph of the first fifty iteration points is shown below. The square symbol represents the initial point, and the vertical and horizontal tick marks are on a scale of 0.1 unit. The orbit of these iteration points appears to spiral into two oscillating points at $-0.78 + 0.18i$ and $-0.21 + -0.18i$.



Iteration points of $Z_0 = -0.8 + 0.1i$

Graph the first fifty iteration points for each of the three given initial points (Z_0). Use a separate sheet of graph paper for each graph. Describe each graph. (Hint: Look at all the real and imaginary coordinate values for the fifty iteration points to determine the best scaling for the graph.)
   a) $Z_0 = 0.35 + 0.2i$
   b) $Z_0 = -0.7 + 0.25i$
   c) $Z_0 = -0.67 + 0.1i$

8. Although the Mandelbrot set is a simple recursive generating function, it was not discovered until the 1970s. On a separate sheet of paper, discuss your thoughts on why it was not discovered for so long, and list some of the most interesting features of the Mandelbrot set.