

# Functions of

# Number Theory in Music

*Mathematics is on the artistic side a creation of new rhythms, orders, designs, harmonies, and on the knowledge side, is a systematic study of various rhythms, orders, designs and harmonies.*

—WILLIAM L. SCHAAFF

*Music is a hidden exercise in arithmetic, of a mind unconscious of dealing in numbers.*

—GOTTFRIED WILHELM VON LEIBNIZ

*They [the Pythagoreans] supposed the whole heaven to be a harmonia and a number.*

—ARISTOTLE

Several fundamental notations of number theory arise in music theory

Connecting mathematics with other disciplines is one of the major goals of many general-education mathematics courses taught at universities and colleges across the nation. Of course, from kindergarten to calculus, good teachers demonstrate and explore these connections as a means of making clear the relevance and importance of mathematics to every person. In particular, I have cheerfully noticed a renewed interest in the relationships between mathematics and music. This article examines several fundamental notions of number theory as they arise in music theory, and in the process, develops several music-related functions. I see an added pedagogical benefit in emphasizing the importance of functions in mathematics with examples from a different arena than the usual function pool. Middle school and high school teachers might find that a subset of these topics are handy nuggets of material that can show examples of applications of mathematics outside the mainstream uses in science and engineering. I created this article as notes for the liberal-arts mathematics course that I teach at my college and have consistently found this topic to be one of the students' favorites.

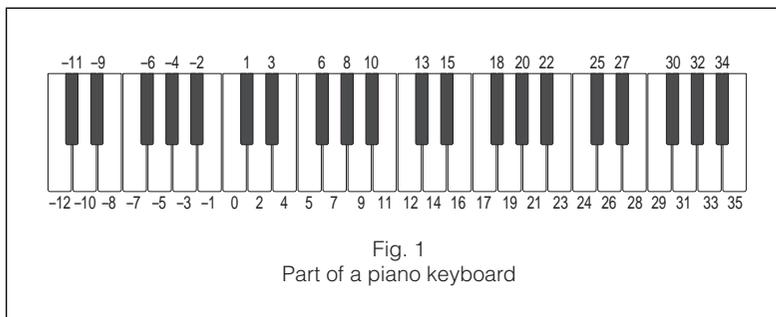
## PRELIMINARY NOTIONS

We notice first that music is the result of an orderly arrangement of different types of sound, which in turn results from waves through the air that cause vibrations of small bones in our ears. The perception of one sound as being either higher or lower than another sound is the result of the specific rapidity of these vibrations, producing a property of sound called *pitch*. The number of vibrations per second is known as the *frequency* of the pitch; and the greater the frequency, the higher the pitch sounds to the ear. We begin by enumerating a piano keyboard, a portion of which is pictured in **figure 1**, as a means of organizing pitch relationships. We call each of these numbers a *key position*.

The interval between any two keys—whether black or white—with consecutive key positions is called a *half step*, or *semitone*, and a basic principle of the keyboard is that any key that is twelve half steps above another key produces a pitch whose frequency is exactly double that of the lower key. To the ear, these two sounds seem related, and they are defined as being one *octave* apart. In **figure 1**, for example, the key in position 4 is one octave higher than the key in position  $-8$ , position 15 is one octave higher than position 3, and position 31 is two octaves higher than position 7. In fact, we see that position  $y$  is  $k$  octaves higher than position  $x$  if

$$y - x = 12k,$$

for  $k = 1, 2, 3, \dots$



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This specific example of a relation between two numbers is known as a *congruence*. Congruence is an important concept in number theory, the study of the properties of integers. Not only does number theory possess an intrinsic beauty equal to that of any other branch of mathematics, but number-theory results often have significant applications in such practical areas as computer science and cryptology. In general, if the difference,  $y - x$ , of two integers is divisible by a positive integer  $m$ , then we say that  $y$  is congruent to  $x$  modulo  $m$  and we write

$$y \equiv x \pmod{m}.$$

The integer  $m$  is called the *modulus*. For example,

$$39 \equiv 4 \pmod{5}$$

because  $39 - 4 = 35$ , which is divisible by 5;

$$109 \equiv 21 \pmod{11}$$

because  $109 - 21 = 88$ , which is divisible by 11;

$$2374 \equiv 374 \pmod{1000}$$

because  $2374 - 374 = 2000$ , which is divisible by 1000;

$$-46 \equiv 10 \pmod{7}$$

because  $-46 - 10 = -56$ , which is divisible by 7.

An equivalent way of defining  $y \equiv x \pmod{m}$  is to say that  $y$  and  $x$  differ by a multiple of  $m$ , that is,  $y - x = mk$ , or

$$y = x + mk,$$

for some integer  $k$ . Given any integer  $x$ , the set of all integers that are congruent to  $x$  modulo  $m$  is called its *residue class*, or *congruence class*, and is denoted by  $[x]_m$ . If  $x$  is any one particular member of a residue class, the remaining members can be generated by adding successive multiples of  $m$  to  $x$ . So, for instance,

$$[2]_6 = \{ \dots, -10, -4, 2, 8, 14, 20, 26, \dots \},$$

$$[3]_7 = \{ \dots, -11, -4, 3, 10, 17, 24, 31, \dots \},$$

$$[5]_{11} = \{ \dots, -17, -6, 5, 16, 27, 38, 49, \dots \}.$$

The least nonnegative integer in the residue class is typically used to represent it and is called the *least residue*. Given any class, the least residue is easily seen to be the remainder left after the particular number is divided by the modulus. For example,  $[748]_5 = [3]_5$  because 3 is the remainder when 748 is divided by 5. The operation of addition of classes is well defined by

$$[a]_m + [b]_m = [a + b]_m.$$

So, for instance,  $[58]_8 + [41]_8 = [99]_8 = [3]_8$ . The use of any other representative in either class would result in the same sum, for example,  $[58]_8 = [2]_8$ ,  $[41]_8 = [1]_8$ , and  $[2]_8 + [1]_8 = [3]_8$ .

## PITCH CLASSES

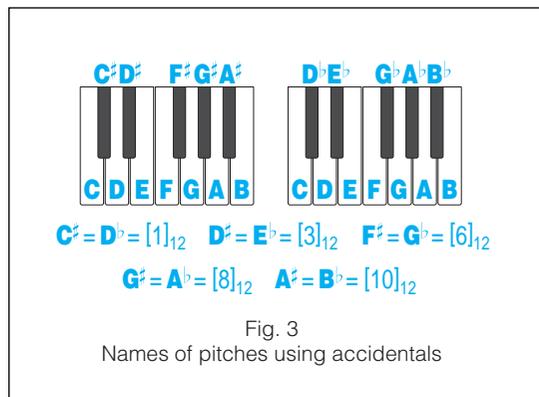
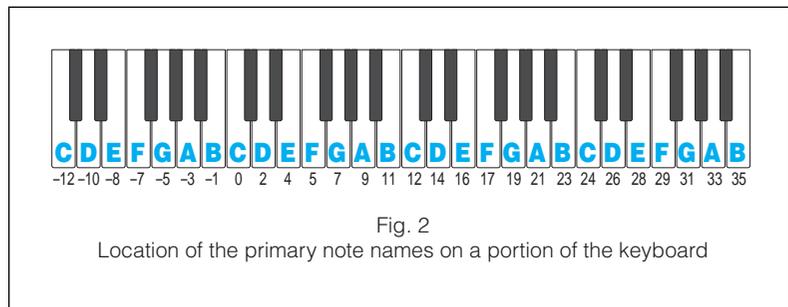
We next turn to the use of congruence in music. Two notes representing pitches that are an integral

multiple of octaves (twelve half steps) apart are designated by the same letter. Each letter, then, is the name of a residue class modulo 12 of key positions. Each of these twelve classes is called a *pitch class*, and the primary note names are defined by the following:

$$\begin{array}{llll} C = [0]_{12} & D = [2]_{12} & E = [4]_{12} & F = [5]_{12} \\ G = [7]_{12} & A = [9]_{12} & B = [11]_{12} & \end{array}$$

Their positions on the keyboard consist of all the white keys, as shown in **figure 2**. An *accidental* following a note name indicates a pitch that is one half step above (sharp) or one half step below (flat) the primary pitch. The names of notes involving accidentals, that is, sharps and flats, are defined as in **figure 3**. Every pitch class can have more than one name. Two note names for the same class are called *enharmonic spellings*.

**Congruence plays a role in both number theory and music**



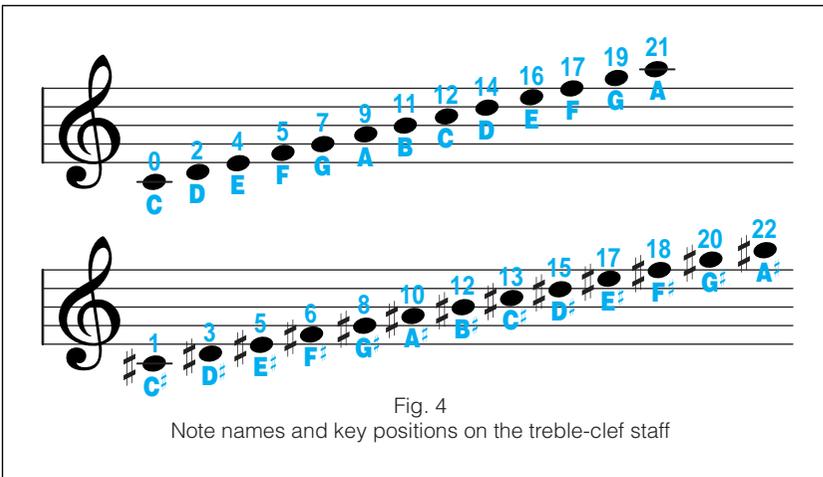
The alphabet of musicians consists of notes, which are ovals placed on a staff. Our note names are the labels of notes on the treble-clef staff, as positioned in **figure 4**.

## THE CIRCLE OF FIFTHS

A major scale is a specific arrangement of eight pitches separated by whole steps, that is, two consecutive half steps, and half steps according to the pattern

whole–whole–half–whole–whole–whole–half.

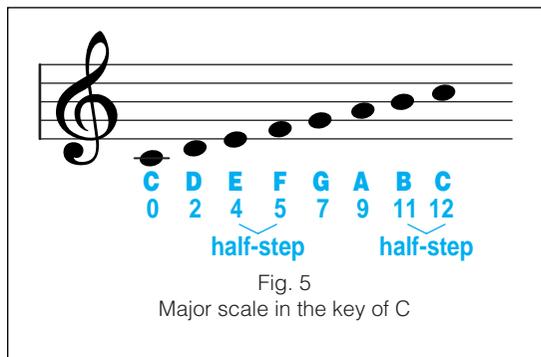
The first note in each major scale is known as the



***In a circle of fifths, each successive pitch is a perfect fifth above the previous one***

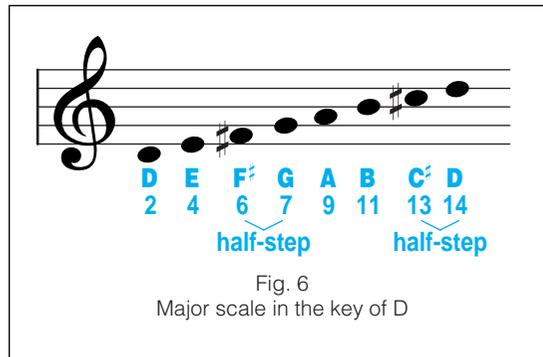
tonic and is called the *key* of that scale. For example, the major scale shown in **figure 5** is in the key of C. All separations in a major scale consist of whole steps except for the half steps between the third and fourth notes and between the seventh and eighth notes. The C-major scale is the only major scale consisting entirely of notes having no accidentals. Every other major scale can simply be thought of as a translation of this basic position sequence:

0, 2, 4, 5, 7, 9, 11, 12

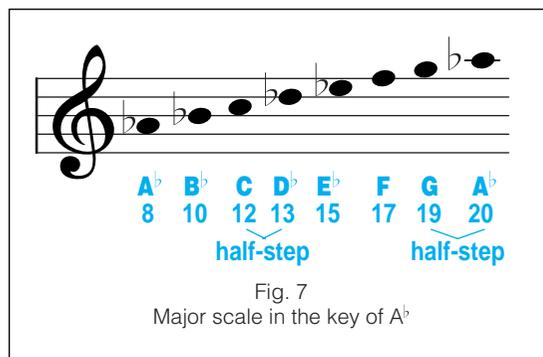


To create a major scale in the key of D, we want a sequence beginning with 2, so we add 2 to each position in the sequence given for the scale of C to obtain 2, 4, 6, 7, 9, 11, 13, 14. See **figure 6**. To maintain the half steps in the correct locations, the two notes F and C are sharped. Similarly, we see that a scale exists for the other possible starting keys, and accidentals must be appended to certain notes in each of these keys to retain the proper structure. For example, since A<sup>b</sup> is located eight positions above C, the corresponding position sequence is

8, 10, 12, 13, 15, 17, 19, 20



and the corresponding scale is pictured in **figure 7**. As a result of the convention that requires the notes of any major scale to be listed in alphabetical order with every letter name appearing exactly once, the accidentals needed in this situation turn out to be flats. The key of any piece of music is identified by the number of sharps or flats placed at the beginning in the key signature. By ordering the major keys according to the number of accidentals in their key signatures, we can find a relationship that can be displayed by using modular arithmetic.



Progressing down the third column in **figure 8**, we see that each successive pitch class is obtained by adding 7 (mod 12). In music theory, a mnemonic device for organizing the preceding relationships is constructed by using a circle, much like a clock that is marked with note names. Starting with the key of C at 12:00 and proceeding clockwise, we mark each successive hour with the major key whose signature contains one more sharp than the preceding key. Likewise, if we proceed counterclockwise from C, we write the key whose signature contains one more flat. The resulting diagram, known as the *circle of fifths*, is shown in **figure 9**. Note that B, F<sup>#</sup>, and C<sup>#</sup> overlap with their enharmonic spellings C<sup>b</sup>, G<sup>b</sup>, and D<sup>b</sup>. The corresponding pitch classes of the keys are marked on the inside of the circle.

This circle obtains its name from the fact that if we proceed clockwise, each successive pitch is a

No. of sharps or flats	Key	Pitch Class
0♯	C	0
1♯	G	7
2♯	D	2
3♯	A	9
4♯	E	4
5♯	B	11
6♯	F♯	6
7♯	C♯	1
7♭	C♭	11
6♭	G♭	6
5♭	D♭	1
4♭	A♭	8
3♭	E♭	3
2♭	B♭	10
1♭	F	5
0♭	C	0

Fig. 8  
Major keys as a function of number and type of accidentals

perfect fifth above the previous one. A *perfect fifth* is a musical interval that consists of five lines or spaces on the staff. Since this distance corresponds to seven half steps, each pitch class can be obtained by adding 7 (mod 12) to the preceding class. This result leads to our first two number-theory functions, each of which determines the major key of a piece of music from its key signature. If  $n$  is the number of sharps in the signature, the key is given by

$$k^\sharp(n) = [7n]_{12}.$$

If  $n$  is the number of flats in the signature, the key is given by

$$k^\flat(n) = [5n]_{12}.$$

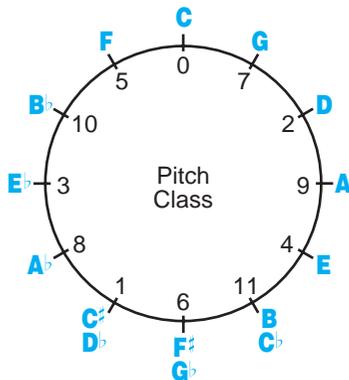


Fig. 9  
The circle of fifths

### Examples



From **figure 6**, we know that the key of D must have two sharps in its signature. This fact is confirmed by  $k^\sharp(2) = [2 \cdot 7]_{12} = [14]_{12} = [2]_{12} = D$ .



The key that has five flats in its signature is  $k^\flat(5) = [5 \cdot 5]_{12} = [25]_{12} = [1]_{12} = D^\flat$ .

### INVERSIONS

Music composition often involves certain transformations of a sequence, or segment, of notes. Most of these transformations have an analog as an isometry in geometry and have been much discussed in musical and mathematical literature. For instance, consider the graphical representation of “The Music of the Night,” by Andrew Lloyd Webber. The opening two measures are given in **figure 10**.

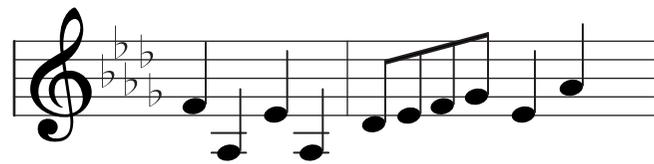


Fig. 10  
“The Music of the Night”

We create a note graph from the corresponding *position sequence*

$$5, -4, 3, -4, 1, 3, 5, 6, 3, 8$$

by plotting it on a set of axes with the sequence order indicated on the horizontal axis and the positions marked on the vertical axis, as shown in **figure 11**. Any musical transformation performed on

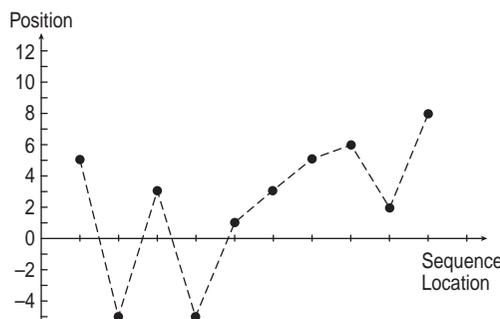
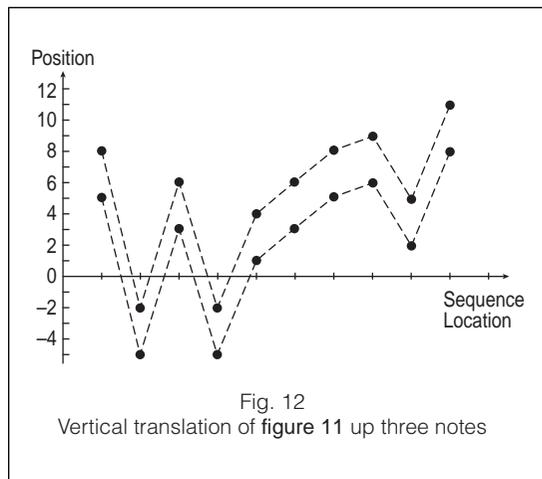


Fig. 11  
Note graph of opening two measures of “The Music of the Night”

**Transforming a sequence of notes has an analog as an isometry in geometry**

**Melodic inversion is similar to the reflection of a mountain range in a lake**



a segment of notes has a corresponding effect on the associated note graph. So, for example, a transposition of a musical segment to a different key is simply a vertical translation of its graph. **Figure 12** demonstrates a vertical translation up three notes. The position sequence

$$5, -4, 3, -4, 1, 3, 5, 6, 3, 8$$

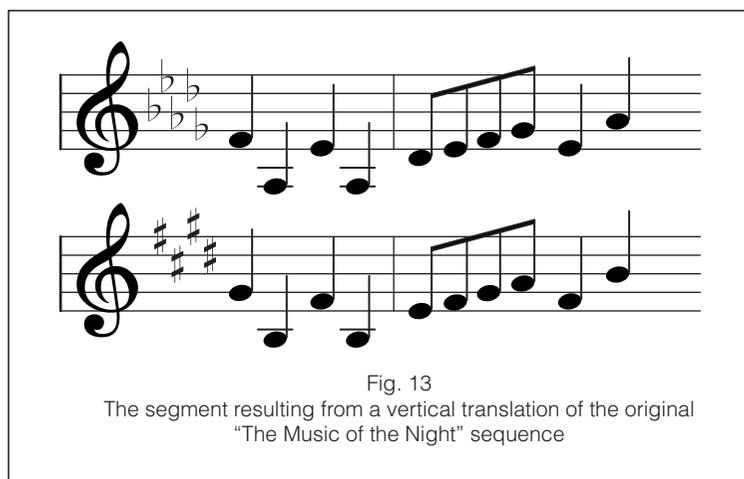
becomes

$$8, -1, 6, -1, 4, 6, 8, 9, 6, 11$$

having note names

$$G^\sharp, B, F^\sharp, B, E, F^\sharp, G^\sharp, A, F^\sharp, B$$

and giving us the new segment of notes shown in **figure 13**. The key has changed from  $D^b = [1]_{12}$  up three notes to the key of  $[4]_{12} = E$ . Other transformations applied to an entire piece can also result in a change of key.



We next turn to the procedure of *melodic inversion*, a technique that results in a mirrorlike imaging of a sequence of notes on the staff. It is similar to the reflection of a mountain range in a lake. It

may be applied strictly or with enough latitude to prevent veering from the key of the piece. In a strict, or real, inversion, the interval lengths between consecutive notes of a specific segment are all exactly preserved, but the direction of each interval is reversed. In a free, or tonal, inversion, the interval progressions are still reversed, but the new notes are required to stay within the tonality of the music. Some interval lengths are then altered by a half step to prevent dissonant sounds.

Free inversion is far more common than strict inversion in tonal music. Certain musical forms, such as fugues and canons, begin with a fundamental musical theme, or subject, which is later repeated in an altered form as an answer to the subject. Modulation and inversion are two of several techniques used to implement such an alteration. Free inversion is exemplified in a passage from J. S. Bach's *The Art of the Fugue* (Nallin 1968, p. 30). The subject is shown in **figure 14a**, and its inversion forms the answer, as shown in **figure 14b**.

The initial note D of **figure 14a** is translated up seven half steps, or a perfect fifth, to the note A of **figure 14b** before beginning the inversion process. The amount of such a translation is arbitrary, and it may be zero. It can be better understood by examining the position sequences of both segments along with the corresponding sequences of interval lengths between successive notes. We find the differences between consecutive positions, as follows:

Subject sequence (a):

$$2, 9, 7, 5, 4, 2, 1, 2, 4, 5, 7, 5, 4, 2$$

Differences in (a):

$$7, -2, -2, -1, -2, -1, 1, 2, \mathbf{1}, \mathbf{2}, -2, -1, -2$$

Answer sequence (b):

$$9, 2, 4, 5, 7, 9, 10, 9, 7, 5, 4, 5, 7, 9$$

Differences in (b):

$$-7, 2, \mathbf{1}, \mathbf{2}, 2, 1, -1, -2, -2, -1, \mathbf{1}, \mathbf{2}, 2$$

The differences in (b) are the negatives of the differences in (a) except for those in boldface. The boldface numbers indicate where Bach substituted the note F for the note  $F^\sharp$  and was therefore using a free inversion rather than a strict one. Had Bach used  $F^\sharp$  instead, the effect would have been an awful assault on the ears and the tonality of the fugue.

As another example, we can contrast the free inversion found in Johannes Brahms's *Intermezzo* in A Major, op. 118, no. 2 (Berry 1966, p. 352) with what would result if it had been done strictly. The subject is given in **figure 15a**; the free inversion, in **figure 15b**; and the strict inversion, in **figure 15c**. The strict inversion looks more complicated than the free one because of the sharps and the double sharp, but this visual complication could be amelio-



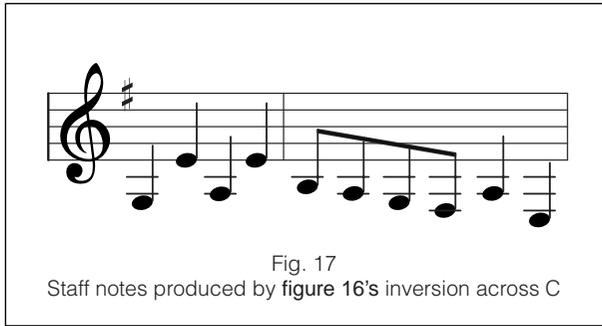


Fig. 17  
Staff notes produced by figure 16's inversion across C

**If we had inverted across a different note, a different key would have resulted**

that is equidistant from 0 but in the opposite direction:

$$-1, -3, -5, -6, -8, -10, -12, -13$$

This result is equivalent modulo 12 to

$$11, 9, 7, 6, 4, 2, 0, (-1)$$

or

$$(1) \quad B, A, G, F^\sharp, E, D, C, B.$$

This scale has one sharp, so by using our previous function, we get  $k^\sharp(1) = [7 \cdot 1]_{12} = [7]_{12} = G$ . So the new key is G, which we observe is the third note in line (1), not the first. The reason is that the new scale in line (1) is descending rather than ascending, so the key is identified by finding the note that is followed by a half step and then three whole steps.

If we had inverted across a note different from C, a different key would have resulted. We previously observed that an inversion across  $p = 0$  could be formulated by

$$n \rightarrow (-n)$$

for each note at position  $n$ . The algebraic effect on  $n$  of an inversion across the pivot  $p$  can be considered the composition of first subtracting  $p$  from  $n$ , that is, translation down to C; followed by negation, that is, inversion across C; and then adding  $p$  back, that is, translation back to the starting point. The net effect can be formulated by

$$n \rightarrow -(n - p) + p,$$

or

$$n \rightarrow 2p - n.$$

This fact leads to the following result.

**THEOREM.** *If an entire piece in the key of  $x$  ( $0 \leq x \leq 11$ ) is inverted across the pivot note with pitch class  $p$ , then the new key,  $I_p(x)$ , is given by*

$$I_p(x) = [2p - x - 4]_{12}.$$

**PROOF.** We must look at the effect of the inversion on the major scale with  $x$  as the tonic note:

$$x \quad x + 2 \quad x + 4 \quad x + 5$$

$$x + 7 \quad x + 9 \quad x + 11 \quad x + 12$$

As in the previous example, the inverted image of the third note,  $x + 4$ , provides the tonic note of the new scale. Therefore, we substitute  $x + 4$  for  $n$  in the formula  $n \rightarrow 2p - n$  to obtain the new key and hence the required function.

**Example**

Let  $p = 0$ . Then  $I_0(x) = [-x - 4]_{12}$ . We have seen the segment in **figure 10** (key of  $D^\flat$ ) inverted to the segment in **figure 17** (key of G). Since  $D^\flat = [1]_{12}$ , this result corresponds to

$$I_0(1) = [-1 - 4]_{12} = [-5]_{12} = [7]_{12} = G.$$

$x$	1	2	3	4	5	6	7	8	9	10	11	0
$I_0(x)$	7	6	5	4	3	2	1	0	11	10	9	8

Fig. 18  
Summary of inversions across C in figures 16 and 17

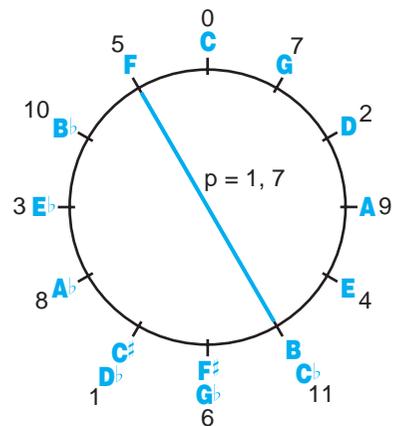
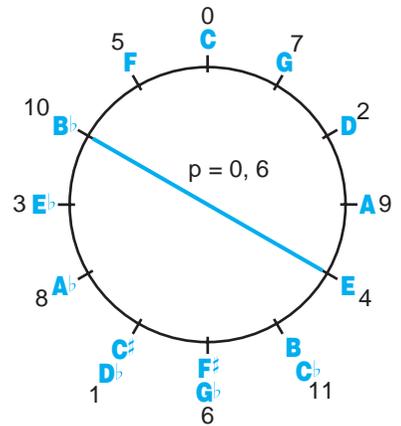


Fig. 19  
Inversions of "The Music of the Night" sequence represented as reflections across the circle of fifths

Similarly, the key of D is inverted across C to

$$I_0(2) = [-2 - 4]_{12} = [-6]_{12} = [6]_{12} = F^\sharp.$$

We summarize these changes in **figure 18**. The keys of E (4) and B<sup>b</sup> (10) are invariant. We observe that the key changes produced by inverting about C can be visualized by a reflection across a line through E and B<sup>b</sup> on the circle of fifths, as shown in the first picture in **figure 19**. Likewise, we can easily verify that the key changes produced by inverting across any other note can be represented by a similar reflection across a line through the invariants. (Question: Why is it true that  $I_{p+6}(x) = I_p(x)$ ?)

## CONCLUSION

When people are asked how mathematics is generally used in the world, their responses usually have to do with the practical side of the discipline related to physics, engineering, business, and similar fields. Rarely do they think of the arts or music, since appreciation and invention within these domains are often considered to spring from the soul. People are always surprised to learn that most mathematicians would heartily agree that the soul is the wellspring of mathematical thought. Indeed, one might argue that the richness of the soul offers the most ideal environment for mathematics to send down its deepest roots. Why? As Edward Rothstein (1995, p. xv) said of his college days, “Music and math together satisfied a sort of abstract ‘appetite,’ a desire that was partly intellectual, partly aesthetic, partly emotional, partly, even, physical.” The occurrence of patterns is the lifeblood of the permeation of mathematics through our existence, and nowhere are the creation and “feel” for patterns more prevalent than in the composition and enjoyment of music.

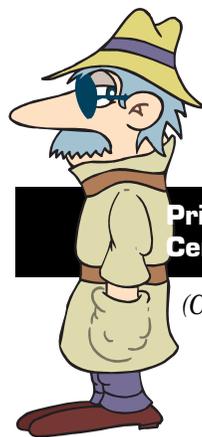
One thing he discovered . . . was that music held more for him than just pleasure. There was meat to it. What the music said was that there is a right way for things to be ordered so that life might not always be just tangle and drift but have a shape, an aim. It was a powerful argument against the notion that things just happen.

—Charles Frazier, *Cold Mountain*

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## Privacy for the Twenty-First Century: Cryptography

(Continued from page 691)

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