ACTIVITIES for students

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Do-It-Yourself Fractal Functions

In the set of fractal activities described in this article, students will accomplish much more than just creating a fun set of cards that simply resemble an art project.

Goals of this activity, designed for an algebra 1 class, are to encourage students to generate data, look for and analyze patterns, and create their own models—all from a set of fractal cards that they construct. The activity and extension could also be used in an algebra 2 class to review exponential equations and practice working with basic exponent rules. These fractal cards produce perfect data for developing concise, explicit equations while requiring students to think critically.

The sequence of activities described in this article can be completed in approximately three class periods. From our experience, the best results are obtained when two or three students work together as a group.

INTRODUCING THE ACTIVITIES

On the first day, introduce (or review) the concept of fractals through pictures (see **fig. 1**) or a video, such as https://www.youtube.com/watch?v=wXBJfaZ2LvU, which hypnotically zooms into the Sierpenski triangle. Then invite students to make observations. Questions similar to the following (about **fig. 1**) may generate discussion:

- What features do these images have in common?
- Look closely at the flower-like designs. How do the flowers (**fig. 1a**) compare to one another?
- Examine the Koch snowflake (**fig. 1b**). How are the various parts of the image related?
- How does the structure of broccoli (**fig. 1c**) relate to the flower and triangle figures?

Through discussion, students begin to understand a fractal as a figure generated through some iterative process by successive subdivisions of a geometric shape into parts. They also notice that the smaller parts of each figure are considered a reduced copy of the larger figure, or whole. A stalk of broccoli is an effective manipulative; cut off one large stem to show that it is similar to the whole head of broccoli and then repeat the process by cutting off a smaller stem. This introduction should take about ten to fifteen minutes of class time to complete.

CUTTING THE CARDS

After discussing fractals, give students four sheets of paper cut to 8 in. × 10 in. (Using sheets of 8 1/2 in. × 11 in. will give the same qualitative results but different values in activity 3.) Precut a set for demonstration purposes so that students can see what they are making (see **photograph 1** on **p. 697**). Four cards seems to be optimal to generate enough data to observe patterns while not becoming too difficult to construct. Ambitious students have successfully created a fifth card. The best approach to the construction of the set of cards is as a teacher-directed activity. We found it helpful to perform the first cuts as a class and then allow students to continue without instruction while we monitor their progress. You will need approximately forty minutes to complete the measuring and cutting process. One advantages to cutting the cards in class is that you can ask these questions while students are engaged:

- What geometric figures or relationships do you see?
- How many edges does a rectangular prism have? Can you find these on your card?
- Can you find a pair of perpendicular lines?

Learner focus: Discovery of fractal patterns and representations; Algebra I, Geometry
Teacher focus: Guide student data generation, pattern recognition, inquiry
Time allotment: 60–90 minutes

Edited by Sam Shah

Activities for Students appears five times each year in *Mathematics Teacher*, promoting reproducible activities adaptable to classroom use. Student-centered activities with teacher facilitation suggestions are welcome. Manuscripts for the department should be submitted via **http://mt.mtsubmit.net**. For more information, visit **http://www.nctm.org/mtcalls**.

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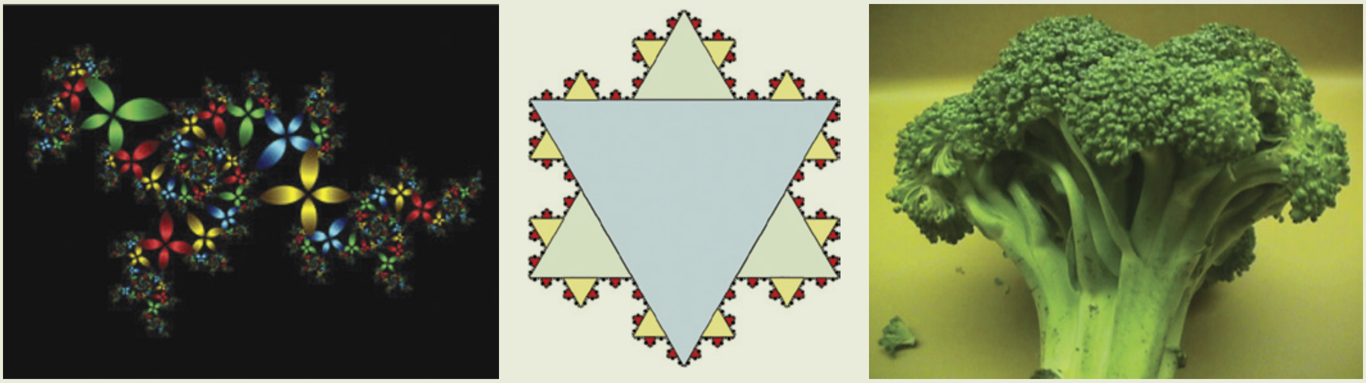

(a) (b) (c)

Fig. 1 Fractals can take various forms: computer generated (a); a Koch snowflake (b); even broccoli (c).

- Which planes are parallel?
- What type of angle is formed between nonparallel planes?
- How many "invisible" planes can you count?

Another advantage of teacher–directed cutting is that it is easy to spot students who have difficulty with measurement and fractions. Some students will struggle with reading a ruler correctly and accurately drawing the line segments that need to be cut. Generally, a quick check with their partners will let them know if they are drawing the lines correctly. To differentiate this part of the lesson, have a set of cards already marked for struggling students to cut.

Much of the fractal activity depends on students' fluency in multiplying a given measurement by 1/2 or 1/4. This is an excellent opportunity to review the multiplication of fractions.

EXAMINING PATTERNS

After all students have constructed the set of four cards, it is time to look for patterns. Ask students to complete a table and answer a set of accompanying questions. Once the data are collected, ask students to try finding equations for both the number of new boxes produced and for the total number of boxes. Students generally have little difficulty giving a recursive description of what is happening in their table: the number of new boxes is double the previous number of boxes. Getting students to write an explicit equation may take more work. Assist students by suggesting that they express the values in column 2 as factors of 2, as in **table 1**. Encourage students to use exponents to express the repeated multiplication of 2. It may be helpful to have students look at the bottom of the table to generalize a pattern in powers of 2. The observation that the exponent is always one less than the stage number will help students express the pattern using a variable for the stage number. Once students realize that the total number of boxes at each stage is one less than the number of boxes added at the following stage, they can use the expression in column 2 to find an expression for column 3.

Table 1 Counted Sequences

Stage or Card Number x	Number of Boxes Added y_1	Total Number of Boxes y_2
1	$1 = 2^0$	1
2	$2 = 2^1$	3
3	$4 = 2^2$	7
4	$8 = 2^3$	15
5	$16 = 2^4$	31
x	$y_1 = 2^{x-1}$	$y_2 = 2^x - 1$

A copy of the student sheet is provided at the end of this article.

BOX PATTERNS FOR FRACTALS

Throughout the activity, we pose questions to our students to help them make connections and tie ideas to other familiar concepts.

- Is your data linear? How do you know?
- How can you be sure that your equation correctly models your data set?
- Does anyone have a different equation for columns 2 and 3?
- How are the equations you found for y_1 and y_2 similar? How are they different?

Once the table is completed, ask students to make predictions using the equations found. Students might be asked, "If it were possible to cut card 10, how many total boxes would be on the card?" or "Let's say you counted 4095 total boxes —what card number would that be?"

An important discussion question involves the generalizations between column 2 and column 3. This is an opportunity to discuss the difference between $2^x - 1$ and 2^{x-1} and focus on the importance of precision (and the importance of writing equations neatly to avoid confusion).

Students who have previously learned how to use a calculator to find regression equations can run an exponential regression on the data in columns 1 and 2. The equation generated, $y = 0.5(2)^x$, provides

Table 2 Attributes That Might Be Counted

What Was Counted	Card 1	Card 2	Card 3	Card 4	Card x
Edges added at each stage	11	22	44	88	$y = 11(2^{x-1})$
Edges added at each stage (including invisible edges)	12	24	48	96	$y = 12(2^{x-1})$
Total edges (including invisible edges)	12	36	84	180	$y = 12(2^x - 1)$
Paper planes added at each stage	2	4	8	16	$y = 2^x$
Total paper planes	2	6	14	30	$y = 2(2^x - 1)$ or $y = 2^{x+1} - 2$
Invisible planes added	4	8	16	32	$y = 4(2^{x-1})$ or $y = 2^{x+1}$
Total invisible planes	4	12	28	60	$y = 4(2^x - 1)$ or $y = 2^{x+2} - 4$
Planes (including the invisible ones) added	6	12	24	48	$y = 6(2^{x-1})$
All planes (including the invisible ones)	6	18	18	90	$y = 6(2^x - 1)$
Vertices added	8	16	32	64	$y = 8(2^{x-1})$ or $y = 2^{x+2}$
Total vertices	8	24	56	120	$y = 8(2^x - 1)$ or $y = 2^{x+3} - 8$
Right angles added each time (no invisible angles).	16	32	64	128	$y = 16(2^{x-1})$ or $y = 2^{x+3}$
Right angles added each time including the invisible angles	24	48	96	192	$y = 24(2^{x-1})$
Total right angle	24	72	168	360	$y = 24(2^x - 1)$
Distance from the edge of the previous rectangle	2	1	½	¼	$y = 2(1/2)^{x-1}$

an opportunity to use algebra to show how it is equivalent to the equation found by hand. The data in columns 1 and 3 generate an equation that can help spark a lively discussion as to why. The calculator regression function always sets the horizontal asymptote to $y = 0$, in contrast to the desired asymptote at $y = -1$.

DISCOVERING NEW PATTERNS

To start the next part of the activity, give students two to three minutes to make a list of all the things they could count or calculate on their fractal cards. These lists can be shared within groups or with the class as a whole. Some students seek out patterns involving angles, areas and volumes while other students simply count planes, edges, or vertices. This activity encourages deep, critical thinking and produces some very interesting patterns. While a few students seek out more complex patterns, most choose patterns that are easily countable.

Students create their own data tables and should be encouraged to mark directly on the cards to aid counting. They discover that although count attributes on stages 1 and 2 is easy, by stage 3 the process becomes tedious and mistakes are easily made. After students have completed stage 3, and perhaps are a bit frustrated, encourage them to look for a pattern to determine stage-4 and stage-5 values instead of trying to continue counting. Looking for and verbalizing these patterns will help students not only continue the pattern, but also generate an explicit equation.

Some of the most common patterns found by our students are included in **table 2**. Notice that some students count the "invisible" parts, such as a missing edge or face, associated with a rectangular prism. After collecting their data, students represent their findings in words, as a table, as a graph, and as an equation and may be asked these questions:

- "Can you describe how you found your algebraic expression?"
- "When your data is graphed in the x-y plane, what is the shape of your graph?"
- "Is your pattern the same or similar to any other group's pattern?"
- "Can you write your equation in a different way?"

A CHALLENGING EXTENSION

The fractal cards can easily be used to enhance and extend students' understanding of functions and patterns. This optional extension can be used to challenge classes that are confident with their algebra skills and have previously been introduced to exponent rules. The answers to activity sheet 3 (**p. 701**) are available in the online **more4U** material.

Some students have difficulties when completing the table in activity 3. Suggest using fractions in columns B and C to help find patterns. To help calculate the areas of the faces, suggest that students lay the cards flat and shade in the new areas added at each stage.

Have students generalize their findings for each column and algebraically explain and support the relationships that they have discovered. **Figure 2** shows a justification for why the ratio of the cut length to the distance measured in for the cut is a constant value of 1.25 (see the table in activity sheet 3, columns B, C, E).

$(5/2^x) \div (1/2^{x-2})$	Write the expression for C ÷ B.
$= (5/2^x) \cdot (2^{x-2})$	Rewrite division as multiplication by the reciprocal.
$= (5 \cdot 2^{-x}) \cdot 2^{x-2}$	Rewrite 2^x from the denominator of the fraction.
$= 5 \cdot 2^{-x+x-2}$	Combine the exponents.
$= 5 \cdot 2^{-2}$	Simplify.
$= 5/4$	Rewrite as a fraction.
$= 1.25$	Rewrite as a decimal.

Fig. 2 Calculations with exponents justify that a particular ratio is constant.

CONCLUSION

The natural appeal of these hands-on cards fascinates learners and engages them with the power of mathematics that advances easily from simple to complex. These fractal explorations can be adapted for varying levels of students who generate their own data; teachers can reinforce vocabulary in a visual, engaging, and concrete manner. The concepts woven into the activities align with NCTM Standards and Common Core State Standards, both of which emphasize rich activities infused with enduring mathematical themes that promote pattern recognition and representation.

REFERENCES

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

Sierpinski Triangle Zoom In. https://www.youtube.com/watch?v=wXBJfaZ2LvU

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more4U

Go to **http://www.nctm.org/mt** for the activity sheets. This online content, an additional benefit, is for members only.

Photograph 1
Three cards

ACTIVITY SHEET 1: CUTTING A SET OF FRACTAL CARDS

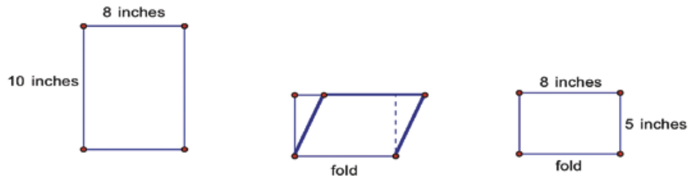
Objective: Experience a repetitive process that, when applied, will create cards exhibiting self-similarity. Each repetition of the process is called an iteration.

Materials: Bright copy paper or construction paper, ruler with inch units, pencil, scissors, four sheets of 8 in. × 10 in. paper.

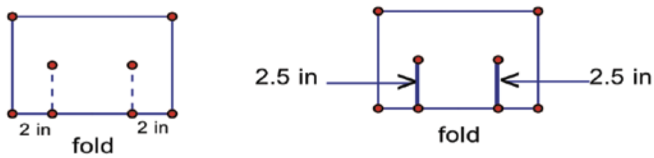
Recommendation: Use colored paper to make each card a different color.

Card 1 (Stage 1 or Iteration 1)

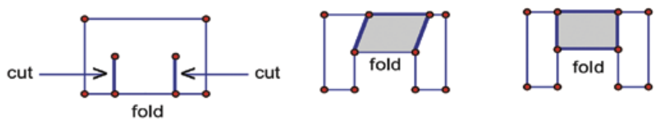
1. Select and fold one of the sheets of paper.



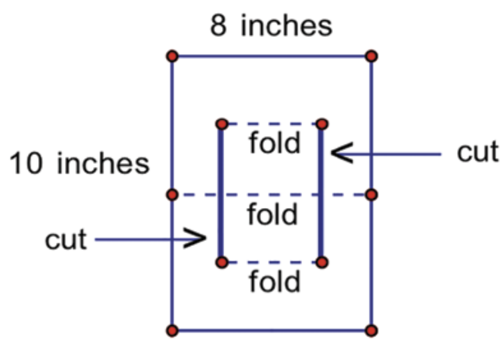
2. Measure in
1/4 • 8 inches = 2 inches.
Draw a segment
1/2 • 5 inches = 2 1/2 inches long.



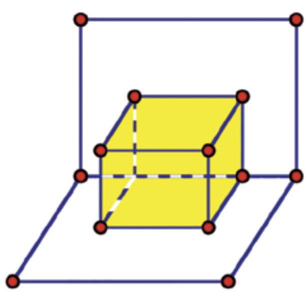
3. Cut and fold.



4. Unfold and crease each fold both ways.



5. Pop out the box to present Card 1.



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Card 2 (Stage 2 or Iteration 2)

1. Select and fold another sheet of paper.

2. As with Card 1, measure in
1/4 • 8 inches = 2 inches. Draw a segment
1/2 • 5 inches = 2 1/2 inches long.

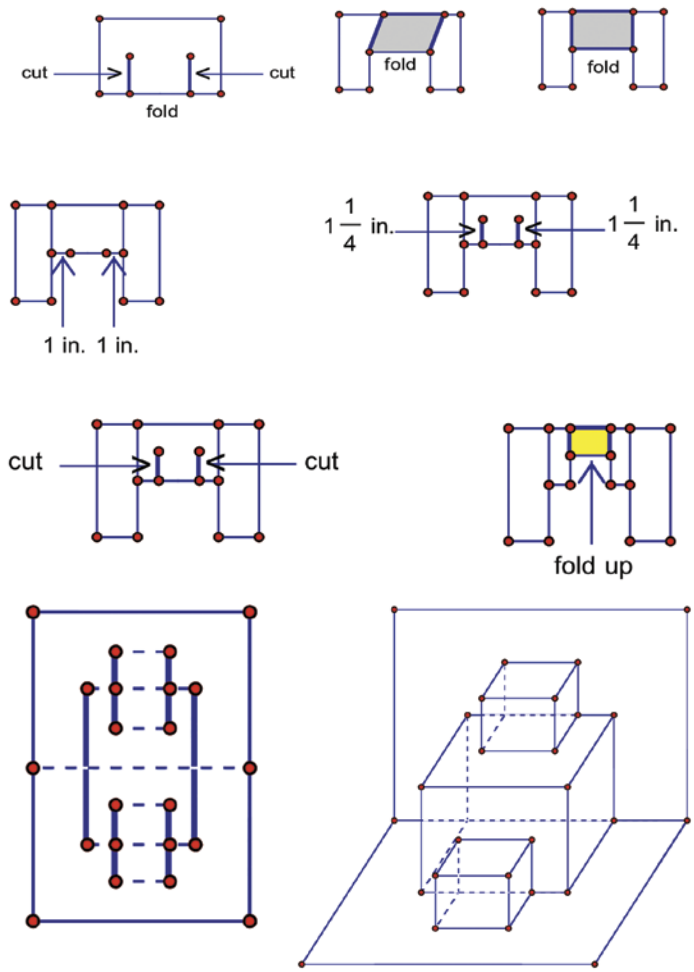
3. As with Card 1, cut and fold. Call the shaded portion the *inner rectangle*.

4. On the inner rectangle, measure in
1/4 • 4 inches = 1 inch. Draw a segment
1/2 • 2 1/2 = 1 1/4 inches long.

5. Cut and fold. The new inner rectangle is shown in yellow.

6. Unfold and crease each fold both ways.

7. Pop out the boxes to present Card 2.



Card 3 (Stage 3 or Iteration 3)

Steps **1**-**5** are the same as for Card 2.

6. Measure in 1/4 • 2 inches = inches. Draw a segment 1/2 • 1 1/4 inches = inches long.

7. Cut and fold.

8. Unfold and crease each fold both ways.

9. Pop out the boxes to present Card 3.

Card 4 (Stage 4 or Iteration 4)

Select your last sheet of paper.

List your steps for cutting Card 4.

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ACTIVITY SHEET 2: BOX PATTERNS FOR FRACTALS

Materials: 2 sheets of graph or grid paper

Objective: Examine and extend patterns relating to the pop-out boxes on the fractal cards.

Use fractal cards 1–4. Then extend the patterns for stages numbered 5 through *9*. Write general expressions for the y-variables in the last row of columns 2 and 3 in terms of the stage-number variable x.

Column 1 **Stage or Card Number,** x	**Column 2** **Number of Boxes Added,** y_1	**Column 3** **Total Number of Boxes,** y_2
1	1	1
2		
3		
4		
5		
6		
7		
8		
9		
x		

1. On graph paper, label the x and y_1 axes. Then use data from Columns 1 and 2 to graph each ordered pair (x, y_1). Describe the shape of the graph.

2. On graph paper, label the x and y_2 axes. Then use data from Columns 1 and 3 to graph each ordered pair (x, y_2). Describe the shape of the graph.

ACTIVITY SHEET 3:
RATIOS AND AREAS FOR FRACTAL CARDS

Materials: Fractal cards

Objective: Examine ratios and areas for the fractal cards.
Use fractal cards 1–4. Extend the patterns to fill in values for the next cards. For the table, look at the newest inner rectangle formed at each stage.

Table Columns:
A: Stage number (card number)
B: In inches, distance to measure for the newest cut
C: In inches, length of the newest cut, written as a fraction in simplest form
D: In inches, length of one face on the newest rectangular prism created
E: The ratio of C to B (i.e., C ÷ B), written as a decimal
F: The area of one face of the smallest box created at the given stage,written as a decimal
G: The area of all box faces at the given stage (the paper covered faces only). Watch out!

Some box faces are part of two different boxes. Do not double count any areas.
The last row of cells should contain a constant value or an expression for the column in terms of x.

A	B	C	D	E	F	G
1	2	2.5 = 5/2	4	1.25	10	20
2						25
3						
4						
x						

1. Use algebra to show why the value in Column E is a constant.
Work on separate paper and show all your steps.

2. Use algebra to show how to simplify the expression in the last row, Column F. Show all your steps.

Challenge: What do you think the values in Column G are approaching? Try to find the exact value.