Students have already worked with solids, finding the volume and surface area of prisms and other shapes built with blocks. Now these skills are extended to finding the volumes and surface areas of pyramids, cones, and spheres.

See the Math Notes boxes in Lessons 11.1.2, 11.1.3, 11.1.4, 11.1.5, and 11.2.2.

Example 1

A regular hexahedron has an edge length of 20 cm. What are the surface area and volume of this solid?

Although the name "regular hexahedron" might sound intimidating, it just refers to a regular solid with six (hexa) faces. As defined earlier, regular means all angles are congruent and all side lengths are congruent. A regular hexahedron is just a cube, so all six faces are congruent squares.

To find the volume of the cube, we can use our previous knowledge: multiply the area of the base by the height. Since the base is a square, its area is 400 square cm. The height is 20 cm, therefore the volume is (400)(20) = 8000 cubic cm.
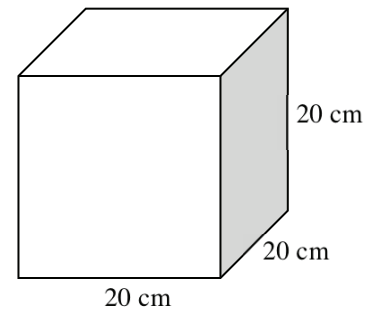
To calculate the surface area we will find the sum of the areas of all six faces. Since each face is a square and they are all congruent, this will be fairly easy. The area of one square is 400 square cm, and there are six of them. Therefore the surface area is 2400 square cm.


20 cm

20 cm

20 cm

Example 2

The base of the pyramid at right is a regular hexagon. Using the measurements provided, calculate the surface area and volume of the pyramid.

The volume of any pyramid is $V = \frac{1}{3} A_b h$ (h is the height of the pyramid and A_b is the area of the base). We calculate the surface area the same way we do for all solids: find the area of each face and base, then add them all together. The lateral faces of the pyramid are all congruent triangles. The base is a regular hexagon. Since we need the area of the hexagon for both the volume and the surface area, we will find it first.

14"

8"
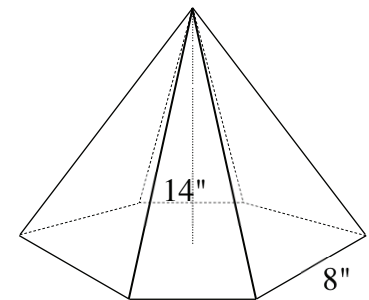
There are several ways to find the area of a regular hexagon. One way is to cut the hexagon into six congruent equilateral triangles, each with a side of 8". If we can find the area of one triangle, then we can multiply by 6 to find the area of the hexagon. To find the area of one triangle we need to find the value of h, the height of the triangle. Recall that we studied these triangles earlier; remember that the height cuts the equilateral triangle into two congruent 30°-60°-90° triangles. To find h, we can use the Pythagorean Theorem, or if you remember the pattern for a 30°-60°-90° triangle, we can use that. With either method we find that $h = 4\sqrt{3}$ ". Therefore the area of one equilateral triangle is shown at right.
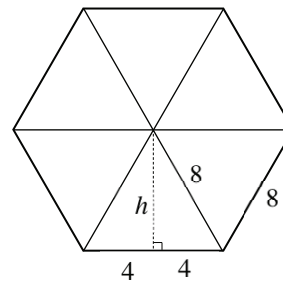
$$A = \tfrac{1}{2}bh$$

$$= \tfrac{1}{2}\cdot(8)\cdot\left(4\sqrt{3}\right)$$

$$= 16\sqrt{3} \approx 27.71 \text{ in.}^2$$

The area of the hexagon is $6\cdot16\sqrt{3} = 96\sqrt{3} \approx 166.28$ in.^{2}. $V = \tfrac{1}{3}A_b h$
Now find the volume of the pyramid using the formula as shown at right.

$$= \tfrac{1}{3}\cdot\left(96\sqrt{3}\right)\cdot(14)$$

$$= 448\sqrt{3} \approx 776 \text{ in.}^3$$

Next we need to find the area of one of the triangular faces. These triangles are slanted, and the height of one of them is called a slant height. The problem does not give us the value of the slant height (labeled c at right), but we can calculate it based on the information we already have.
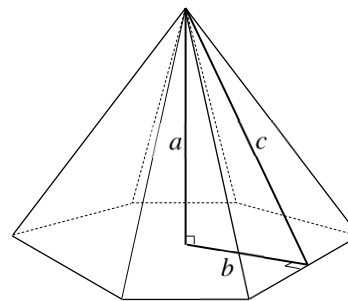
A cross section of the pyramid at right shows a right triangle in its interior. One leg is labeled a, another b, and the hypotenuse c. The original picture gives us $a = 14$". The length of b we found previously: it is the height of one of the equilateral triangles in the hexagonal base. Therefore, $b = 4\sqrt{3}$ ". To calculate c, we use the Pythagorean Theorem.

$$a^2 + b^2 = c^2$$

$$14^2 + (4\sqrt{3})^2 = c^2$$

$$196 + 48 = c^2$$

$$c^2 = 244$$

$$c = \sqrt{244} = 2\sqrt{61} \approx 15.62 \text{ "}$$

The base of one of the slanted triangles is 8", the length of the side of the hexagon. Therefore the area of one slanted triangle is $8\sqrt{61} \approx 62.48$ in.^{2} as shown below right.
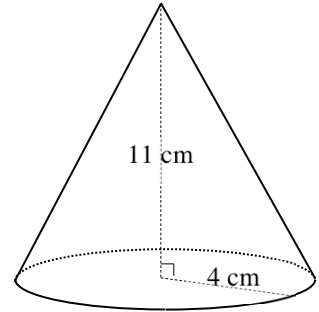
$$A = \tfrac{1}{2}\cdot b\cdot h$$

$$= \tfrac{1}{2}\cdot(8)\cdot\left(2\sqrt{61}\right)$$

Since there are six of these triangles, the area of the lateral faces is $6(8\sqrt{61}) = 48\sqrt{61} \approx 374.89$ in.^{2}.

$$= 8\sqrt{61} \approx 62.48 \text{ in.}^2$$

Now we have all we need to find the total surface area: $96\sqrt{3} + 48\sqrt{61} \approx 541.17$ in.^{2}.
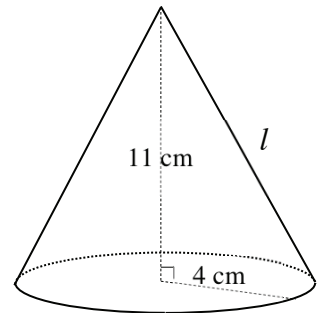
Example 3

The cone at right has the measurements shown. What are the lateral surface area and volume of the cone?

The volume of a cone is the same as the volume of any pyramid: $V = \frac{1}{3} A_b h$. The only difference is that the base is a circle, but since we know how to find the area of a circle ($A = \pi r^2$), we find the volume as shown at right.
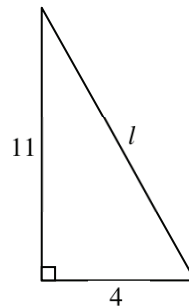
$$V = \frac{1}{3} A_b h$$
$$= \frac{1}{3}\left(\pi r^2\right) h$$
$$= \frac{1}{3}\left(\pi \cdot 4^2\right) \cdot 11$$
$$= \frac{1}{3} \cdot (176\pi) = \frac{176\pi}{3}$$
$$\approx 184.3 \text{ cm}^3$$

Calculating the lateral surface area of a cone is a different matter. If we think of a cone as a child's party hat, we can imagine cutting it apart to make it lay flat. If we did, we would find that the cone is really a sector of a circle – not the circle that makes up the base of the cone, but a circle whose radius is the slant height of the cone. By using ratios we can come up with the formula for the lateral surface area of the cone, $SA = \pi r l$, where r is the radius of the base and l is the slant height. In this problem, we have r, but we do not have l. Find it by taking a cross section of the cone to create a right triangle. The legs of the right triangle are 11 cm and 4 cm, and l is the hypotenuse. Using the Pythagorean Theorem we can calculate $l \approx 11.7$ cm, as shown below right.

$$4^2 + 11^2 = l^2$$
$$16 + 121 = l^2$$
$$l^2 = 137$$
$$l = \sqrt{137} \approx 11.7 \text{ cm}$$

Now we can calculate the lateral surface area:
$$SA = \pi(4)(11.7) \approx 147.1 \text{ cm}^2$$

Example 4

The sphere at right has a radius of 6 feet. Calculate the surface area and the volume of the sphere.

Since spheres are related to circles, we should expect that the formulas for the surface area and volume will have π in them. The surface area of a sphere with radius r is $4\pi r^2$. Since we know the radius of the sphere is 6, $SA = 4\pi(6)^2 = 144\pi \approx 452.39$ ft^2. To find the volume of the sphere, we use the formula $V = \frac{4}{3}\pi r^3$. Therefore, $V = \frac{4}{3}\pi(6)^3 = \frac{4 \cdot 216 \cdot \pi}{3} = 288\pi \approx 904.78$ ft^3.
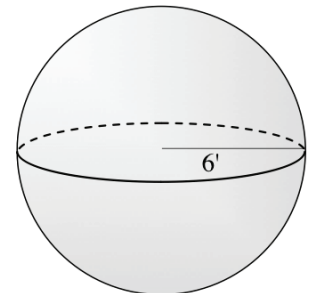
Problems

1. The figure at right is a square based pyramid. Calculate its surface area and its volume.

2. Another pyramid, congruent to the one in the previous problem, is glued to the bottom of the first pyramid, so that their bases coincide. What is the name of the new solid? Calculate the surface area and volume of the new solid.
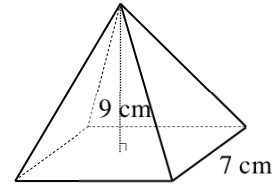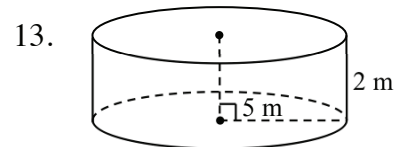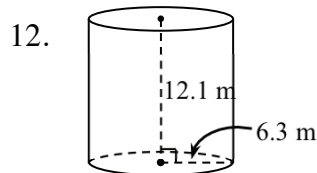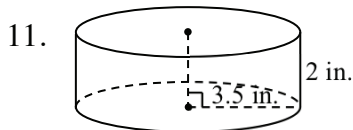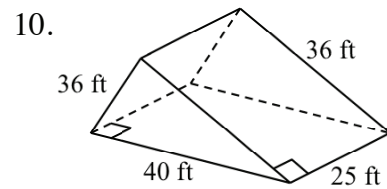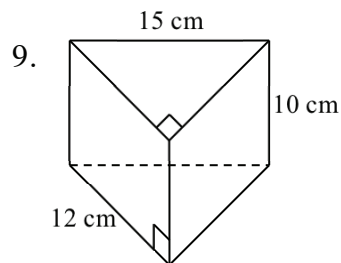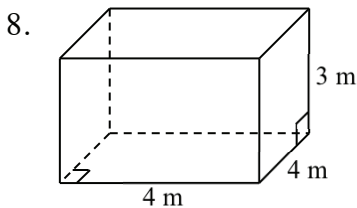
3. A regular pentagon has a side length of 10 in. Calculate the area of the pentagon.

4. The pentagon of the previous problem is the base of a right pyramid with a height of 18 in. What is the surface area and volume of the pyramid?
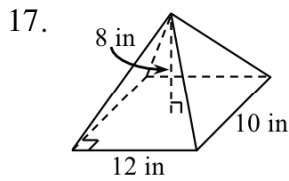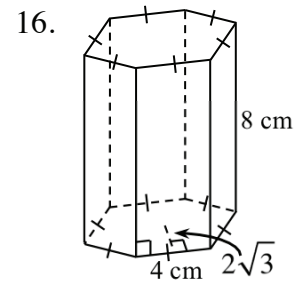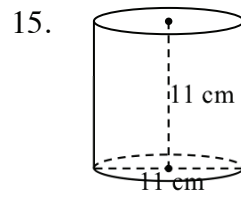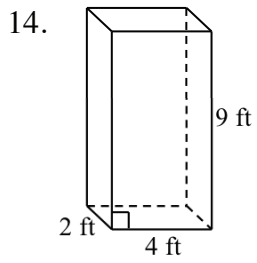
5. What is the total surface area and volume of the cone at right?

6. A cone fits perfectly inside a cylinder as shown. If the volume of the cylinder is 81π cubic units, what is the volume of the cone?
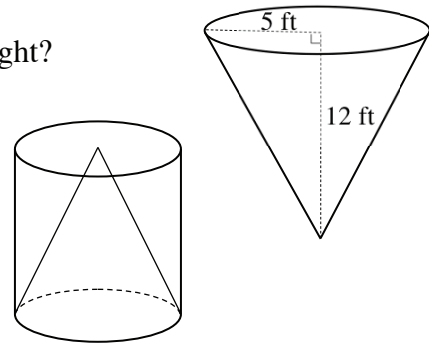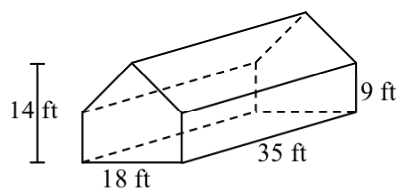
7. A sphere has a radius of 12 cm. What are the surface area and volume of the sphere?
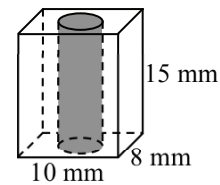
Find the volume of each figure.

8.

9.

10.

11.

12.

13.

14.

9 ft

2 ft

4 ft

15.

11 cm

11 cm

16.

8 cm

4 cm $2\sqrt{3}$

17.

8 in

10 in

12 in

18.

5 in

13 in

19.

11 cm

8 cm

20.

1.0 m

2.6 m

21.

10 in.

16 in.

16 in.

22.

11 m

18 m

23.

6.1 cm

6.2 cm

8.3 cm

24.

5 cm

2 cm

25.

7 in.

8 in.

8 in.
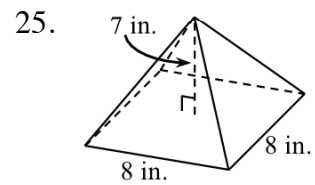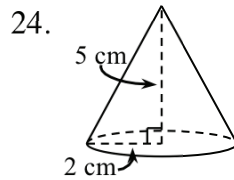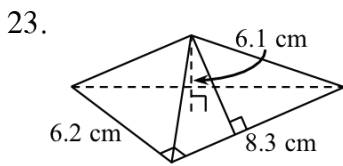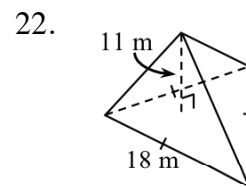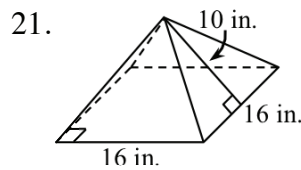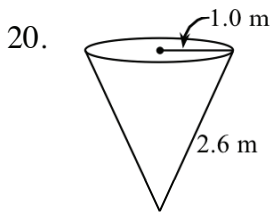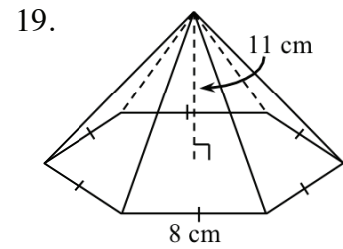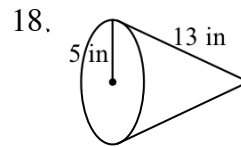
26. Find the volume of the solid shown.

14 ft

9 ft

35 ft

18 ft

27. Find the volume of the remaining solid after a hole with a diameter of 4 mm is drilled through it.

15 mm

10 mm   8 mm

Find the total surface area of the figures in the previous volume problems.

28. Problem 1

29. Problem 2

30. Problem 3

31. Problem 5

32. Problem 6

33. Problem 7

34. Problem 8

35. Problem 9

36. Problem 10

37. Problem 14

38. Problem 18

39. Problem 19

Use the given information to find the volume of the cone.

40. radius = 1.5 in.
    height = 4 in.

41. diameter = 6 cm
    height = 5 cm

42. base area = 25π
    height = 3

43. base circum. = 12π
    height = 10

44. diameter = 12
    slant height = 10

45. lateral area = 12π
    radius = 1.5

Use the given information to find the lateral area of the cone.

46. radius = 8 in.
    slant height = 1.75 in.

47. slant height = 10 cm
    height = 8 cm

48. base area = 25π
    slant height = 6

49. radius = 8 cm
    height = 15 cm

50. volume = 100π
    height = 5

51. volume = 36π
    radius = 3

Use the given information to find the volume of the sphere.

52. radius = 10 cm

53. diameter = 10 cm

54. circumference of
    great circle = 12π

55. surface area = 256π

56. circumference of
    great circle = 20 cm

57. surface area = 100

Use the given information to find the surface area of the sphere.

58. radius = 5 in.

59. diameter = 12 in.

60. circumference of
    great circle = 14

61. volume = 250

62. circumference of
   great circle = π

63. volume = $\frac{9\pi}{2}$

Answers

1. $V = 147 \text{ cm}^3$, $SA \approx 184.19 \text{ cm}^2$

2. Octahedron, $V = 294 \text{ cm}^3$, $SA \approx 270.38 \text{ cm}^2$

3. $A \approx 172.05 \text{ in.}^2$

4. $V = 1032.29 \text{ in.}^3$, $SA \approx 653.75 \text{ in.}^2$

5. $V \approx 314.16 \text{ ft}^3$, $SA = 90\pi \approx 282.74 \text{ ft}^2$

6. 27π cubic units

7. $SA = 576\pi \approx 1089.56 \text{ cm}^2$, $V = 2304\pi \approx 7238.23 \text{ cm}^3$

8. 48 m^3

9. 540 cm^3

10. 14966.6 ft^3

11. 76.9 m^3

12. 1508.75 m^3

13. 157 m^3

14. 72 ft^3

15. 1045.4 cm^3

16. 332.6 cm^3

17. 320 in.^3

18. 314.2 in.^3

19. 609.7 cm^3

20. 2.5 m^3

21. 512 m^3

22. 514.4 m^3

23. 2.3 cm^3

24. 20.9 cm^3

25. 149.3 in.^3

26. 7245 ft^3

27. 1011.6 mm^3

28. 80 m^2

29. 468 cm^2

30. 3997.33 ft^2

31. 727.98 m^2

32. $50\pi + 20\pi \approx 219.8 \text{ m}^2$

33. 124 ft^2

34. $121\pi + 189.97 \approx 569.91 \text{ cm}^2$

35. $192 + 48\sqrt{3} \approx 275.14 \text{ cm}^2$

36. 213.21 in.^2

37. 576 in.^2

38. 193.0 in.^2

39. 2394.69 ft^2

40. $3\pi \approx 9.42 \text{ in.}^3$

41. $15\pi \approx 47.12 \text{ cm}^3$

42. $25\pi \approx 78.54 \text{ units}^3$

43. $120\pi \approx 376.99 \text{ units}^3$

44. $96\pi \approx 301.59 \text{ units}^3$

45. $\approx 18.51 \text{ units}^3$

46. $14\pi \approx 43.98 \text{ in.}^2$

47. $60\pi \approx 188.50 \text{ cm}^2$

48. $30\pi \approx 94.25 \text{ units}^2$

49. $136\pi \approx 427.26 \text{ cm}^2$

50. $\approx 224.35 \text{ units}^2$

51. 116.58 units^2

52. $\frac{4000\pi}{3} \approx 4188.79 \text{ cm}^3$

53. $\frac{500\pi}{3} \approx 523.60 \text{ cm}^3$

54. $288\pi \approx 904.79 \text{ units}^3$

55. $\frac{2048\pi}{3} \approx 2144.66 \text{ units}^3$

56. $\approx 135.09 \text{ units}^3$

57. $\approx 94.03 \text{ units}^3$

58. $100\pi \approx 314.16 \text{ units}^2$

59. $144\pi \approx 452.39 \text{ untis}^2$

60. $\approx 62.39 \text{ units}^2$

61. $\approx 191.91 \text{ units}^2$

62. $\pi \approx 3.14 \text{ units}^2$

63. $9\pi \approx 28.27 \text{ units}^2$

Throughout the year, students have varied their study between two-dimensional objects to three-dimensional objects. This section applies these studies to geometry on a globe. Students learn terms associated with a globe (longitude, latitude, equator, great circle), how the globe is divided, and how to locate cities on it. Additionally, they can find the distance between two cities with the same latitude. Students also notice that some of the facts that are true on flat surfaces change on a curved surface. For instance, it is possible to have a triangle with two right angles on a sphere.

Example 1

If Annapolis, Maryland is at approximately 75° west of prime meridian, and 38° north of the equator, and Sacramento, California is approximately 122° west of prime meridian, and 38° north of the equator, approximate the distance between the two cities. (The Earth's radius is approximately 4000 miles.)
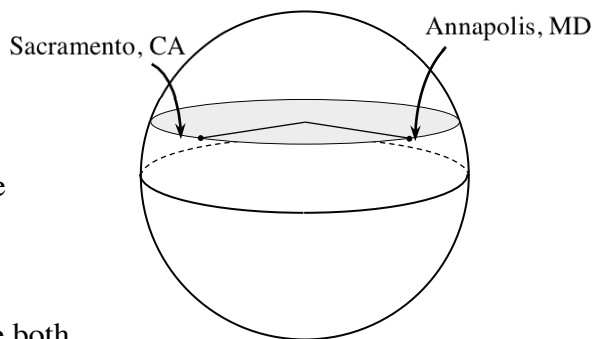
The two cities lie on the same latitude, so they are both on the circumference of the shaded circle. The central angle that connects the two cities is $122° - 75° = 47°$. This means that the arc length between the two cities is $\frac{47}{360}$ of the circle's circumference. To find the shaded circle's circumference, we must find the radius of the shaded circle.
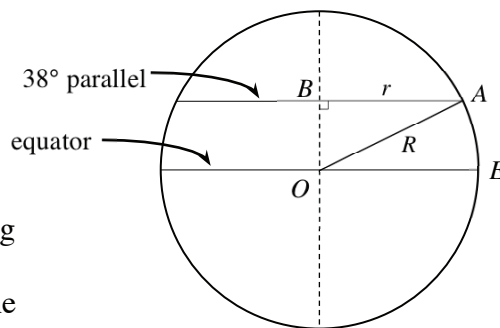
Looking at a cross section of the globe we see something familiar: triangles. In the diagram R is the radius of the Earth while r is the radius of the shaded circle. Since the shaded circle is at 38° north, $m\angle EOA = 38°$. Because the latitude lines are parallel, we also know that $m\angle BAO = 38°$.

We use trigonometry to solve for r, as shown at right. This is the radius of the circle on which the two cities lie. Next we find the fraction of its circumference that is the distance, D, between the two cities.

$$\cos 38° = \frac{r}{R}$$
$$\cos 38° = \frac{r}{4000}$$
$$r = 4000 \cos 38°$$
$$r \approx 3152$$

Therefore the cities are approximately 2586 miles apart.

$$D = \frac{47}{360}(2 \cdot \pi \cdot r)$$
$$= \frac{47}{360}(2 \cdot \pi \cdot 3152)$$
$$\approx 2586 \text{ miles}$$

Problems

1. Lisbon, Portugal is also 38° north of the equator, but it is 9° west of the prime meridian. How far is Annapolis, MD from Lisbon?

2. How far is Sacramento, CA from Lisbon?

3. Port Elizabeth, South Africa is about 32° south of the equator and 25° east of the prime meridian. Perth, Australia is also about 32° south, but 115° east of the prime meridian. How far apart are Port Elizabeth and Perth?

Answers

1. ≈ 3631 miles.

2. ≈ 6216 miles

3. ≈ 5328 miles

In this lesson, students consider lengths of segments and measures of angles formed when tangents and secants intersect inside and outside of a circle. Recall that a tangent is a line that intersects the circle at exactly one point. A secant is a line that intersects the circle at two points. As before, the explanations and justifications for the concepts are dependent on triangles.

See the Math Notes box in Lesson 11.2.3.

Example 1

In the circle at right, $m\widehat{IY} = 60°$ and $m\widehat{NE} = 40°$. What is $m\angle IPY$?

The two lines, $\overleftrightarrow{IE}$ and $\overleftrightarrow{YN}$, are secants since they each intersect the circle at two points. When two secants intersect in the interior of the circle, the measures of the angles formed are each one-half the sum of the measures of the intercepted arcs. Hence $m\angle IPY = \frac{1}{2}(m\widehat{IY} + m\widehat{NE})$ since $\widehat{IY}$ and $\widehat{NE}$ are the intercepted arcs for this angle. Therefore:

$$m\angle IPY = \tfrac{1}{2}(m\widehat{IY} + m\widehat{NE})$$
$$= \tfrac{1}{2}(60° + 40°)$$
$$= 50°$$
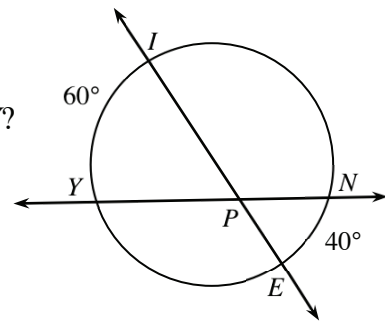
Example 2

In the circle at right, $m\widehat{OA} = 140°$ and $m\widehat{RH} = 32°$. What is $m\angle OCA$?

This time the secants intersect outside the circle at point C. When this happens, the measure of the angle is one-half the difference of the measures of the intercepted arcs. Therefore:

$$m\angle OCA = \tfrac{1}{2}(m\widehat{OA} - m\widehat{RH})$$
$$= \tfrac{1}{2}(140° - 32°)$$
$$= 54°$$
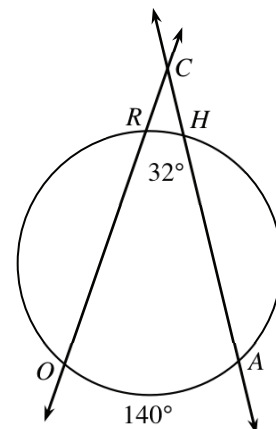
Example 3

$\overrightarrow{MI}$ and $\overrightarrow{MK}$ are tangent to the circle. $m\widehat{ILK} = 199°$ and $MI = 13$. Calculate $m\widehat{IK}$, $m\angle IMK$, and the length of $\overline{MK}$.

When tangents intersect a circle we have a similar result as we did with the secants. Here, the measure of the angle is again one-half the difference of the measures of the intercepted arcs. But before we can find the measure of the angle, we first need to find $m\widehat{IK}$. Remember that there are a total of 360° in a circle, and here the circle is broken into just two arcs. If $m\widehat{ILK} = 199°$, then $m\widehat{IK} = 360° - 199° = 161°$. Now we can find $m\angle IMK$ by following the steps shown at right.

$$m\angle IMK = \tfrac{1}{2}(m\widehat{ILK} - m\widehat{IK})$$
$$= \tfrac{1}{2}(199° - 161°)$$
$$= 19°$$

Lastly, when two tangents intersect, the segments from the point of intersection to the point of tangency are congruent. Therefore, $MK = 13$.

Example 4

In the figure at right, $DO = 20$, $NO = 6$, and $NU = 8$. Calculate the length of $\overline{UT}$.

We have already looked at what happens when secants intersect inside the circle. (We did this when we considered the lengths of parts of intersecting chords. The chord was just a portion of the secant. See the Math Notes box in Lesson 10.1.4.) Now we have the secants intersecting outside the circle. When this happens, we can write $NO \cdot ND = NU \cdot NT$. In this example, we do not know the length of $\overline{UT}$, but we do know that $NT = NU + UT$. Therefore we can write and solve the equation at right.
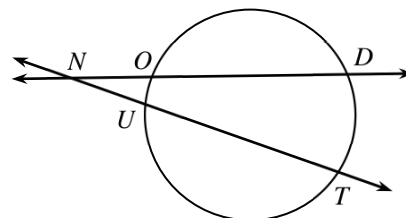
$$NO \cdot ND = NU \cdot NT$$
$$6 \cdot (6 + 20) = 8 \cdot (8 + UT)$$
$$156 = 64 + 8UT$$
$$92 = 8UT$$
$$UT = 11.5$$

Problems

In each circle, C is the center and $\overline{AB}$ is tangent to the circle point B. Find the area of each circle.

1. $\overline{AC} = 30$

2. $\overline{AC} = 45$

3. $6\sqrt{3}$, $30°$

4. 18 B, $45°$

5. 12, $\overline{AC} = 16$

6. $\overline{AC} = 86$, 56

7. 28 B, $\overline{AC} = 90$

8. 24, $\overline{AD} = 18$
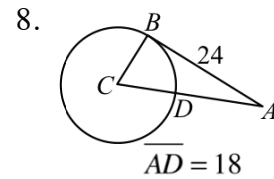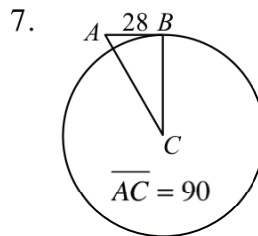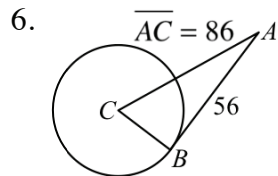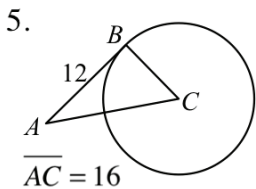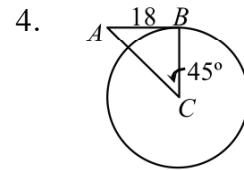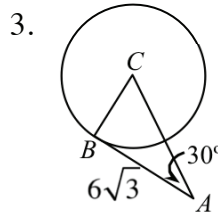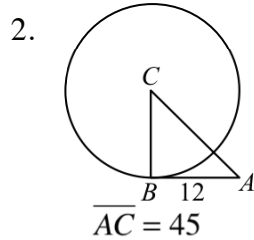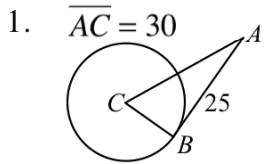
9. In the figure at right, point E is the center and $m\angle CED = 55°$. What is the area of the circle?
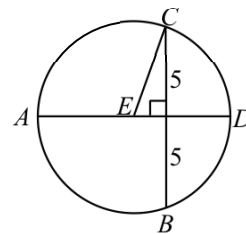
In the following problems, B is the center of the circle.
Find the length of $\overline{BF}$ given the lengths below.

10. $EC = 14, AB = 16$

11. $EC = 35, AB = 21$

12. $FD = 5, EF = 10$

13. $EF = 9, FD = 6$
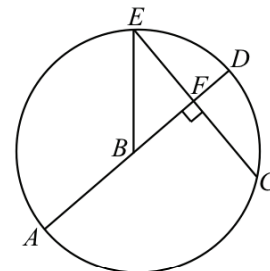
14. In $\odot R$, if $AB = 2x - 7$ and $CD = 5x - 22$, find x.
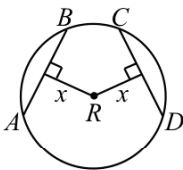
15. In $\odot O$, $\overline{MN} \cong \overline{PQ}$, $MN = 7x + 13$, and $PQ = 10x - 8$. Find PS.
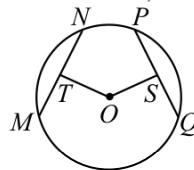
16. In $\odot D$, if $AD = 5$ and $TB = 2$, find AT.
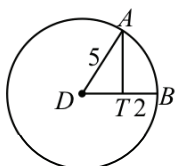
17. In $\odot J$, radius JL and chord MN have lengths of 10 cm. Find the distance from J to $\overline{MN}$.
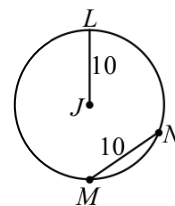
18. In $\odot O$, $OC = 13$ and $OT = 5$. Find AB.
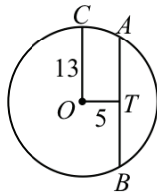
19. If $\overline{AC}$ is tangent to circle E and $\overline{EH} \perp \overline{GI}$, is $\triangle GEH \sim \triangle AEB$? Prove your answer.
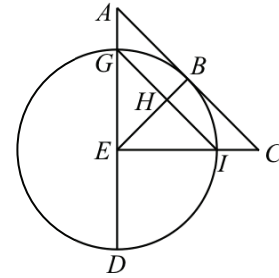
20. If $\overline{EH}$ bisects $\overline{GI}$ and $\overline{AC}$ is tangent to circle E at point B, are $\overline{AC}$ and $\overline{GI}$ parallel? Prove your answer.

Compute the value of x.

21.

22.

23.

24.
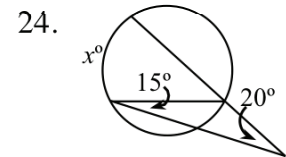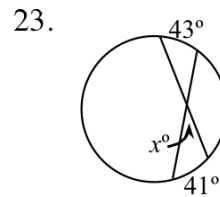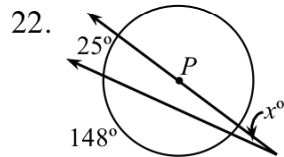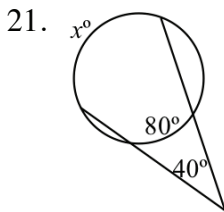
In $\odot F$, $m\overarc{AB} = 84°$, $m\overarc{BC} = 38°$, $m\overarc{CD} = 64°$, $m\overarc{DE} = 60°$. Find the measure of each angle and arc.

25. $m\overarc{EA}$

26. $m\overarc{AEB}$

27. $m\angle 1$

28. $m\angle 2$

29. $m\angle 3$

30. $m\angle 4$
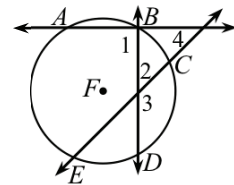
31. If $m\overarc{ADC} = 212°$, what is $m\angle AEC$?

32. If $m\overarc{AB} = 47°$ and $m\angle AED = 47°$, what is $m\overarc{AD}$?

33. If $m\overarc{ADC} = 3 \cdot m\overarc{AC}$ what is $m\angle AEC$?

34. If $m\overarc{AB} = 60°$, $m\overarc{AD} = 130°$, and $m\overarc{DC} = 110°$, what is $m\angle DEC$?

35. If $\overrightarrow{RN}$ is a tangent, $RO = 3$, and $RC = 12$, what is the length of $\overline{RN}$?
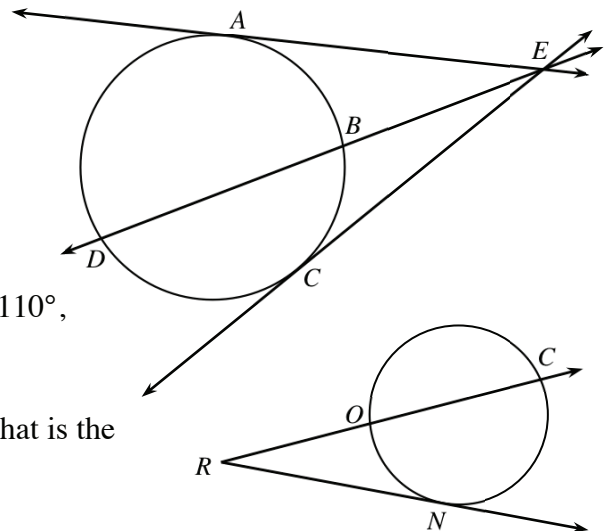
36. If $\overleftarrow{RN}$ is a tangent, $RC = 4x$, $RO = x$, and $RN = 6$, what is the length of $\overline{RC}$?

37. If $\overrightarrow{LT}$ is a tangent, $LU = 16$, $LN = 5$, and $LA = 6$, what are the lengths of $\overline{LW}$ and $\overline{NU}$?
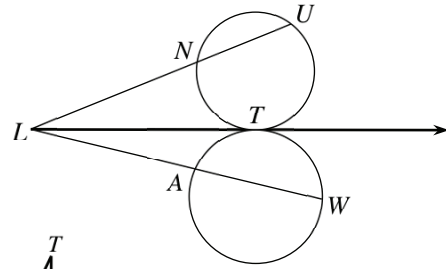
38. If $\overrightarrow{TY}$ is a tangent, $BT = 20$, $UT = 4$, and $AT = 6$, what is the length of $\overline{EA}$ and $\overline{BE}$?

Answers

1. 275π sq. units

2. 1881π sq. units

3. 36π sq. units

4. 324π sq. units

5. 112π sq. units

6. 4260π sq. units

7. 7316π sq. units

8. 49π sq. units

9. ≈ 117.047 sq. units

10. ≈ 14.4

11. ≈ 11.6

12. ≈ 7.5

13. 3.75

14. 5

15. 31

16. 4

17. $5\sqrt{3}$ cm

18. 24

19. Yes, $\angle GEH \cong \angle AEB$ (reflexive). $\overline{EB}$ is perpendicular to $\overline{AC}$ since it is tangent so $\angle GHE \cong \angle ABE$ because all right angles are congruent. So the triangles are similar by AA~.

20. Yes. Since $\overline{EH}$ bisects $\overline{GI}$ it is also perpendicular to it (SSS). Since $\overline{AC}$ is a tangent, $\angle ABE$ is a right angle. So the lines are parallel since the corresponding angles are right angles and all right angles are equal.

21. 160   22. 9   23. 42   24. 70   25. 114

26. 276   27. 87   28. 49   29. 131   30. 38

31. 32°   32. 141°   33. 90°   34. 25°   35. 6

36. 12   37. $LW = \frac{40}{3}$ and $NU = 11$ 38. $EA = \frac{22}{3}$ and $BE = \frac{20}{3}$