WARMUP - Copy these formulas into your notes

Antiderivatives of Trig Functions

1) $\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$
2) $\int \csc\theta \, d\theta = -\ln|\csc\theta + \cot\theta| + C$
3) $\int \sin\theta \, d\theta = -\cos\theta + C$
4) $\int \cos\theta \, d\theta = \sin\theta + C$
5) $\int \sec^2\theta \, d\theta = \tan\theta + C$
6) $\int \csc^2\theta \, d\theta = -\cot\theta + C$
7) $\int \tan\theta \, d\theta = -\ln|\cos\theta| + C$
8) $\int \cot\theta \, d\theta = \ln|\sin\theta| + C$
9) $\int \sec\theta \tan\theta \, d\theta = \sec\theta + C$
10) $\int \csc\theta \cot\theta \, d\theta = -\csc\theta + C$

Useful Derivatives

$$\frac{d}{d\theta}[\tan\theta] = \sec^2\theta$$

$$\frac{d}{d\theta}[\sec\theta] = \sec\theta\tan\theta$$

Pythagorean Identities

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\sec^2\theta = 1 + \tan^2\theta$$

$$\tan^2\theta = \sec^2\theta - 1$$

$$\frac{1}{\sin\theta} = \csc\theta \qquad \frac{1}{\cos\theta} = \sec\theta$$

$$\sin\theta = \frac{opp}{hyp} \qquad \csc\theta = \frac{hyp}{opp}$$

$$\cos\theta = \frac{adj}{hyp} \qquad \sec\theta = \frac{hyp}{adj}$$

$$\tan\theta = \frac{opp}{adj} \qquad \cot\theta = \frac{adj}{opp}$$

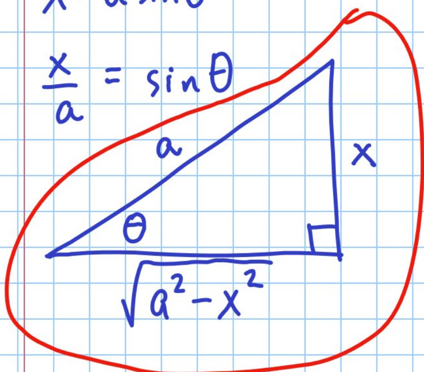
TRIG SUBSTITUTION

CASE 1: For integrals involving $\sqrt{a^2-x^2}$, let $\boxed{x = a\sin\theta}$

$$\sqrt{a^2-x^2} = \sqrt{a^2-(a\sin\theta)^2} = \sqrt{a^2-a^2\sin^2\theta}$$
$$= \sqrt{a^2(1-\sin^2\theta)}$$
$$= \sqrt{a^2\cos^2\theta}$$

$x = a\sin\theta$

$\frac{x}{a} = \sin\theta$



$$\boxed{\sqrt{a^2-x^2} = a\cos\theta}$$

$adj^2 + x^2 = a^2$

$adj^2 = a^2 - x^2$

$adj = \sqrt{a^2-x^2}$

ex: $\displaystyle\int \frac{1}{x^2\sqrt{9-x^2}}\,dx$ $\qquad x = 3\sin\theta$

$\sqrt{3^2-x^2}$ $\qquad dx = 3\cos\theta\, d\theta$

$\sqrt{9-x^2} = 3\cos\theta$

$$= \int \frac{1}{(3\sin\theta)^2 \cdot \cancel{3\cos\theta}} \cdot \cancel{3\cos\theta}\, d\theta$$

$$= \int \frac{1}{9\sin^2\theta}\, d\theta$$

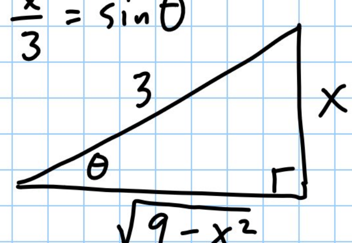
$$= \frac{1}{9}\int \frac{1}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{9}\int \csc^2\theta\, d\theta$$

$$= \frac{1}{9}(-\cot\theta) + C$$

$x = 3\sin\theta$

$\frac{x}{3} = \sin\theta$

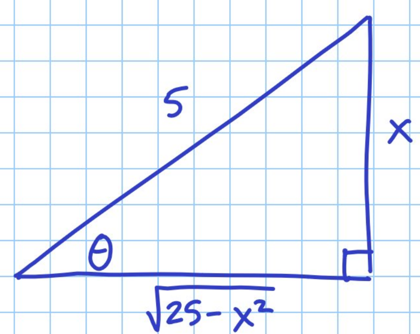


$$= -\frac{1}{9}\cot\theta + C = -\frac{1}{9}\cdot\frac{\overset{adj}{\sqrt{9-x^2}}}{\underset{opp}{x}} + C = -\frac{\sqrt{9-x^2}}{9x} + C$$

Now try:

$$\int \frac{1}{x\sqrt{25-x^2}}\,dx \qquad \begin{aligned} x &= 5\sin\theta \\ dx &= 5\cos\theta\, d\theta \\ \sqrt{25-x^2} &= 5\cos\theta \end{aligned}$$

$$= \int \frac{1}{5\sin\theta \cdot \cancel{5\cos\theta}} \cdot \cancel{5\cos\theta}\, d\theta$$



$$= \frac{1}{5}\int \frac{1}{\sin\theta}\, d\theta$$

$$= \frac{1}{5}\int \csc\theta\, d\theta$$

$$= \frac{1}{5}\left(-\ln\left|\csc\theta + \cot\theta\right|\right) + C$$

$$= -\frac{1}{5}\ln\left|\frac{5}{x} + \frac{\sqrt{25-x^2}}{x}\right| + C$$

$$= -\frac{1}{5}\ln\left|\frac{5+\sqrt{25-x^2}}{x}\right| + C$$

CASE 2: For integrals involving $\sqrt{a^2+x^2}$ (or $\sqrt{x^2+a^2}$)

let $\boxed{x = a\tan\theta}$

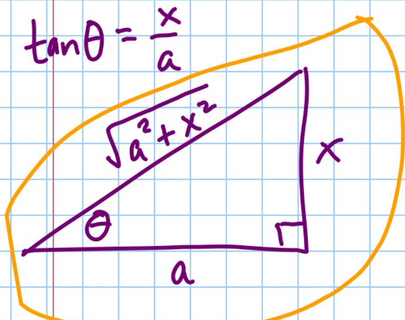
$$\sqrt{a^2+x^2} = \sqrt{a^2+(a\tan\theta)^2} = \sqrt{a^2+a^2\tan^2\theta}$$

$$= \sqrt{a^2(1+\tan^2\theta)}$$

$$= \sqrt{a^2\sec^2\theta}$$

$x = a\tan\theta$

$\tan\theta = \frac{x}{a}$



$$\boxed{\sqrt{a^2+x^2} = a\sec\theta}$$

ex: $\displaystyle\int \frac{1}{(16+x^2)^{3/2}}\,dx$ $\quad x = 4\tan\theta$ ← nowhere to put this

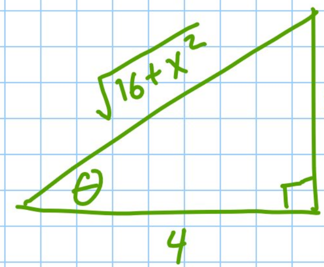
$4^2 + x^2$ $\quad dx = \boxed{4\sec^2\theta\,d\theta}$

$\sqrt{16+x^2} = 4\sec\theta$

$$\int \frac{1}{\left(\sqrt{16+x^2}\right)^3}\,dx$$

$$\int \frac{1}{(4\sec\theta)^3}\cdot 4\sec^2\theta\,d\theta$$

$$\int \frac{4\cancel{\sec^2\theta}}{64\sec^{\cancel{3}}\theta}\,d\theta$$



$x = 4\tan\theta$

$\frac{x}{4} = \tan\theta$

$$\frac{1}{16}\int \frac{1}{\sec\theta}\,d\theta$$

$$\frac{1}{16}\int \cos\theta\,d\theta$$

$$\frac{1}{16}\sin\theta + C = \frac{1}{16}\cdot\frac{x}{\sqrt{16+x^2}} + C = \boxed{\frac{x}{16\sqrt{16+x^2}} + C}$$

$$\int \frac{x}{\sqrt{49-x^2}}\,dx$$