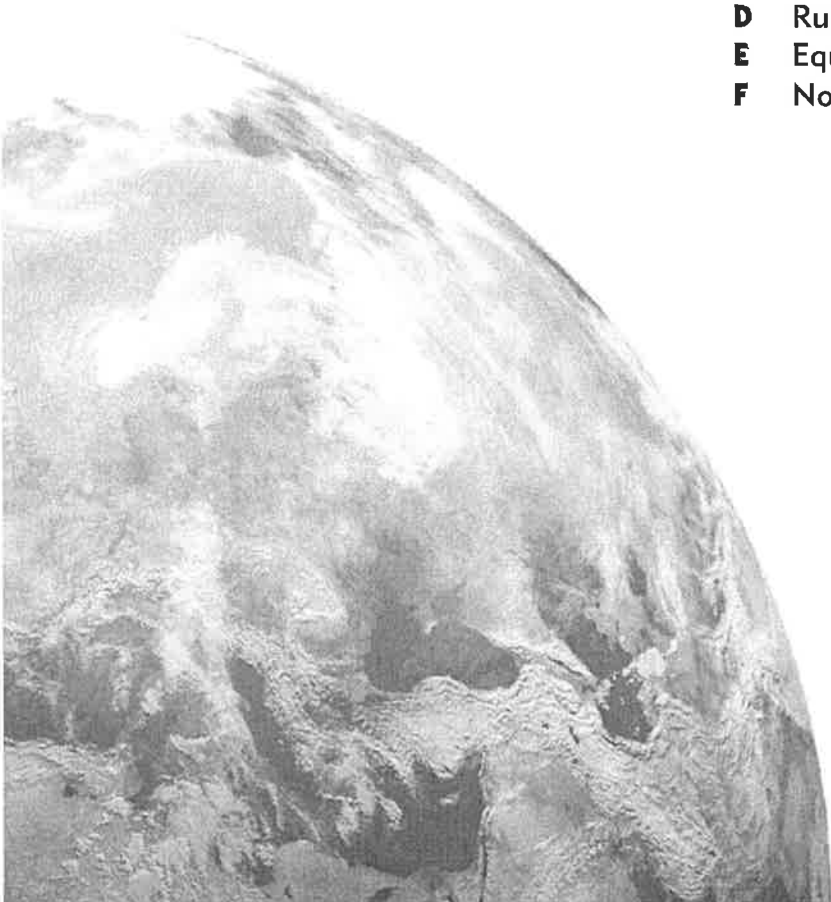
with ANSWERS

Chapter 20

Differential calculus

Syllabus reference: 7.1, 7.2, 7.3

Contents:



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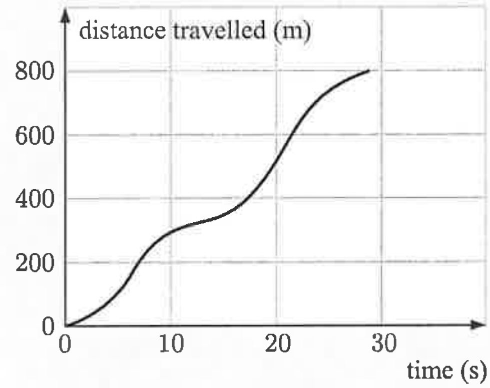
OPENING PROBLEM

Valentino is riding his motorbike around a racetrack. A computer chip on his bike measures the distance Valentino has travelled as time goes on. This data is used to plot a graph of Valentino's progress.



Things to think about:

a What is meant by a *rate*?

b What do we call the rate at which Valentino is travelling?

c What is the difference between an *instantaneous* rate and an *average* rate?

d How can we read a rate from a graph?

e How can we identify the fastest part of the racetrack?

A RATES OF CHANGE

A **rate** is a comparison between two quantities of different kinds.

Rates are used every day to measure performance.

For example, we measure:

- the **speed** at which a car is travelling in km h^{-1} or m s^{-1}.
- the **fuel efficiency** of a car in km L^{-1} or litres per 100 km travelled.
- the **scoring rate** of a basketballer in points per game.

Example 1 — Self Tutor

Josef typed 213 words in 3 minutes and made 6 errors, whereas Marie typed 260 words in 4 minutes and made 7 errors. Compare their performance using rates.

$$\text{Josef's typing rate} = \frac{213 \text{ words}}{3 \text{ minutes}} = 71 \text{ words per minute.}$$

$$\text{Josef's error rate} = \frac{6 \text{ errors}}{213 \text{ words}} \approx 0.0282 \text{ errors per word.}$$

$$\text{Marie's typing rate} = \frac{260 \text{ words}}{4 \text{ minutes}} = 65 \text{ words per minute.}$$

$$\text{Marie's error rate} = \frac{7 \text{ errors}}{260 \text{ words}} \approx 0.0269 \text{ errors per word.}$$

$\therefore$ Josef typed at a faster rate, but Marie typed with greater accuracy.

EXERCISE 20A.1

1 Karsten's pulse rate was measured at 67 beats per minute.

a Explain exactly what this rate means.

b How many heart beats would Karsten expect to have each hour?

2 Jana typed a 14 page document and made eight errors. If an average page of typing has 380 words, find Jana's error rate in:

a errors per word **b** errors per 100 words.

3 Niko worked 12 hours for $148.20, whereas Marita worked 13 hours for $157.95. Who worked for the better hourly rate of pay?

4 New tyres have a tread depth of 8 mm. After driving for 32 178 km, the tread depth on Joanne's tyres was reduced to 2.3 mm. What was the wearing rate of the tyres in:

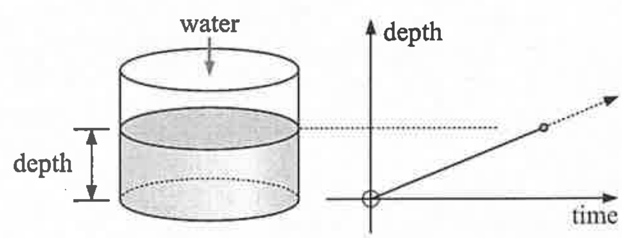
a mm per km travelled **b** mm per 10 000 km travelled?

5 We left Kuala Lumpur at 11:43 am and travelled to Penang, a distance of 350 km. We arrived there at 3:39 pm. What was our average speed in: **a** km h^{-1} **b** m s^{-1}?

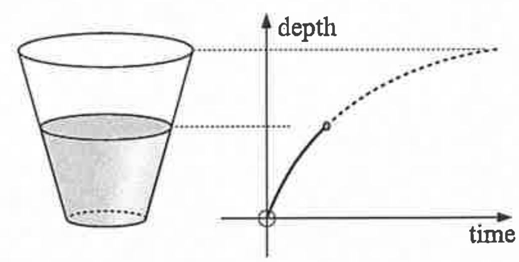
INVESTIGATION 1 CONSTANT AND VARIABLE RATES OF CHANGE

When water is added at a **constant rate** to a cylindrical container, the depth of water in the container is a linear function of time.

The depth-time graph for a cylindrical container is shown alongside.



In this investigation we explore the changes in the graph for different shaped containers such as a conical vase.

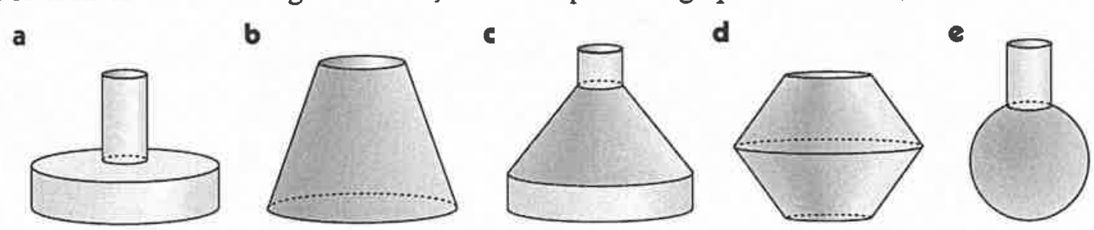


What to do:

1 What features of the graph indicate a rate of change in water level that is:

a constant **b** variable?

2 For each of the following containers, draw a depth-time graph as water is added:



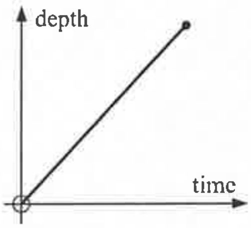
Use the water filling demonstration to check your answers.

DEMO

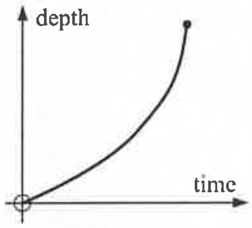
3 Write a brief report on the connection between the shape of a vessel and the shape of its depth-time graph. You may wish to discuss this in parts. For example, first examine cylindrical containers, then conical, then other shapes.

4 Suggest possible container shapes that will have the following depth-time graphs:

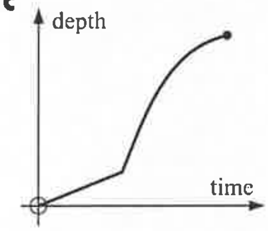
a



b



c



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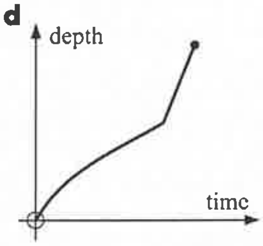


AVERAGE RATE OF CHANGE

If the graph which compares two quantities is a **straight line**, there is a constant rate of change in one quantity with respect to the other. This constant rate is the gradient of the straight line.

If the graph is a **curve**, we can find the **average rate of change** between two points by finding the gradient of the chord or line segment between them. The average rate of change will vary depending on which two points are chosen, so it makes sense to talk about the average rate of change over a particular interval.

Example 2 **Self Tutor**

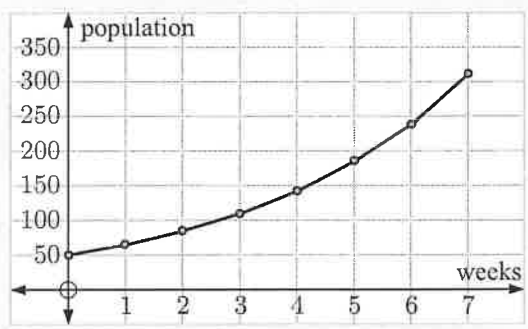
The number of mice in a colony was recorded on a weekly basis.

a Estimate the average rate of increase in population for:

  i the period from week 3 to week 6

  ii the seven week period.

b What is the overall trend in the population growth over this period?



a **i** population growth rate

$$= \frac{\text{increase in population}}{\text{increase in time}}$$

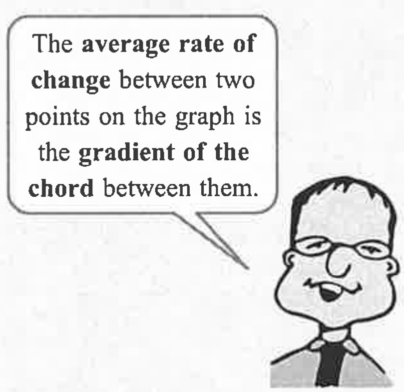
$$= \frac{(240 - 110)\text{ mice}}{(6 - 3)\text{ weeks}}$$

≈ 43 mice per week

ii population growth rate

$$= \frac{(315 - 50)\text{ mice}}{(7 - 0)\text{ weeks}}$$

≈ 38 mice per week

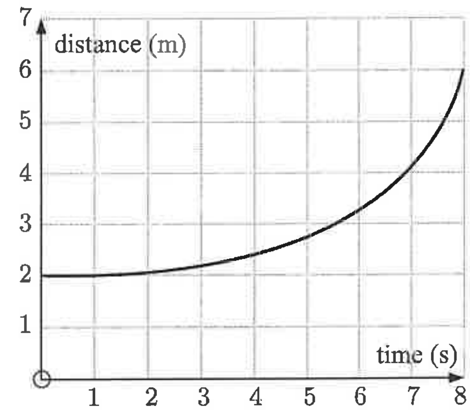


b The graph is increasing over the period by larger and larger amounts, so the population is increasing at an ever increasing rate.

EXERCISE 20A.2

1 For the travel graph given alongside, estimate the average speed:

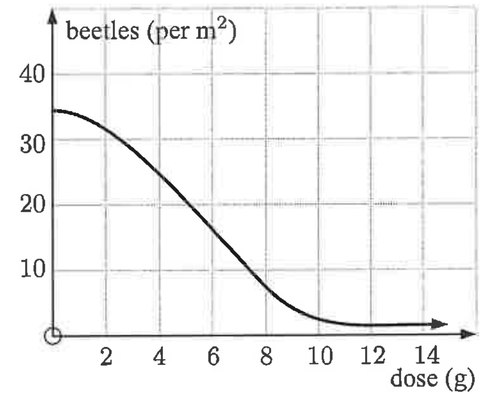
a in the first 4 seconds
b in the last 4 seconds
c in the 8 second interval.



2 The numbers of surviving beetles per m^2 of lawn after various doses of poison, are shown in the graph alongside.

a Estimate the rate of beetle decrease when:
  i the dose increases from 0 to 10 g
  ii the dose increases from 4 to 8 g.

b Describe the effect on the rate of beetle decline as the dose goes from 0 to 14 g.

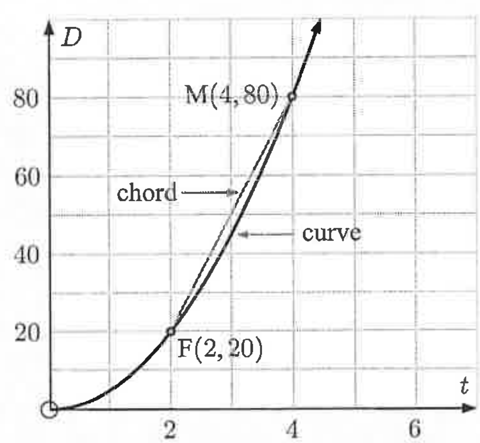


B INSTANTANEOUS RATES OF CHANGE

The speed of a moving object such as a motor car, an aeroplane, or a runner, will vary over time. The speed of the object at a particular instant in time is called its **instantaneous speed**. We examine this concept in the following investigation.

INVESTIGATION 2 INSTANTANEOUS SPEED

A ball bearing is dropped from the top of a tall building. The distance it has fallen after t seconds is recorded, and the following graph of distance against time obtained.



The *average* speed in the time interval $2 \leqslant t \leqslant 4$ is

$$= \frac{\text{distance travelled}}{\text{time taken}}$$

$$= \frac{(80-20)\text{ m}}{(4-2)\text{ s}}$$

$$= \tfrac{60}{2}\text{ m s}^{-1}$$

$$= 30\text{ m s}^{-1}$$

However, this does not tell us the *instantaneous* speed at any particular time.

In this investigation we will try to measure the speed of the ball at the instant when $t = 2$ seconds.

What to do:

1 Click on the icon to start the demonstration.
F is the point where $t = 2$ seconds, and M is another point on the curve.
To start with, M is at $t = 4$ seconds.
The number in the box marked *gradient* is the gradient of the chord FM. This is the *average speed* of the ball bearing in the interval from F to M. For M at $t = 4$ seconds, you should see the average speed is $30\ \text{m s}^{-1}$.

DEMO

2 Click on M and drag it slowly towards F. Copy and complete the table alongside with the gradient of the chord FM for various times t.

t	gradient of FM
3	
2.5	
2.1	
2.01	

3 Observe what happens as M reaches F. Explain why this is so.

4 When $t = 2$ seconds, what do you suspect the instantaneous speed of the ball bearing is?

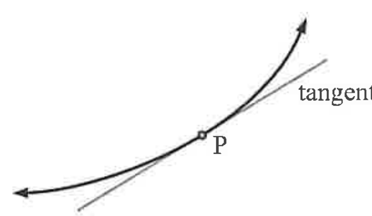
5 Move M to the origin, and then slide it towards F from the left. Copy and complete the table alongside with the gradient of the chord FM for various times t.

t	gradient of FM
0	
1.5	
1.9	
1.99	

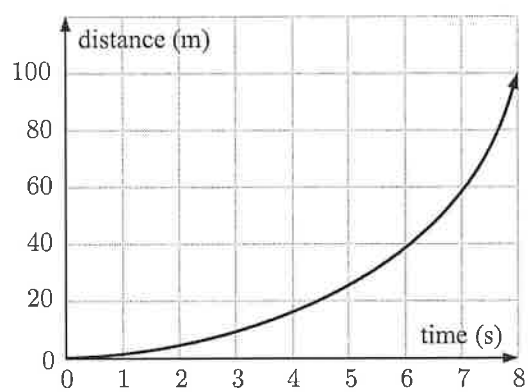
6 Do your results agree with those in **4**?

From the investigation you should have discovered that:

The **instantaneous rate of change** of a variable at a particular instant is given by the **gradient of the tangent** to the graph at that point.



For example, the graph alongside shows how a cyclist accelerates away from an intersection.

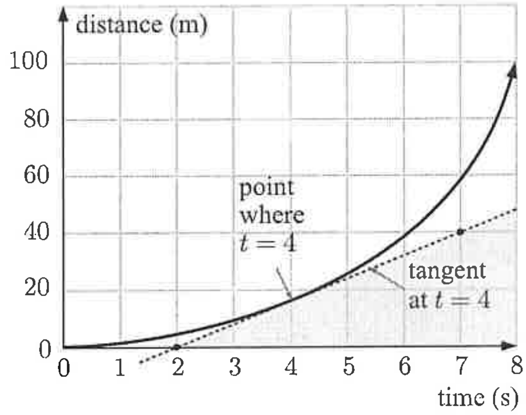


The average speed over the first 8 seconds is

$$\frac{100\ \text{m}}{8\ \text{sec}} = 12.5\ \text{m s}^{-1}.$$

Notice that the cyclist's early speed is quite small, but it increases as time goes by.

To find the instantaneous speed at any time instant, for example $t = 4$, we draw the tangent to the graph at that time and find its gradient.



The tangent passes through (2, 0) and (7, 40).

$\therefore$ the instantaneous speed at $t = 4$

$= $ the gradient of the tangent

$= \dfrac{(40 - 0)\ \text{m}}{(7 - 2)\ \text{s}}$

$= \frac{40}{5}\ \text{m s}^{-1}$

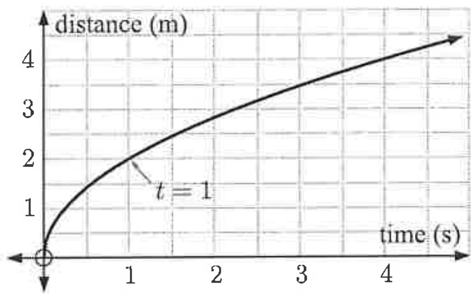
$= 8\ \text{m s}^{-1}$

EXERCISE 20B.1

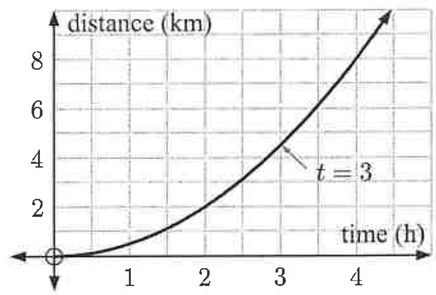
PRINTABLE GRAPHS

1 For each of the following graphs, estimate the rate of change at the point shown by the arrow. Make sure your answer has the correct units.

a



b



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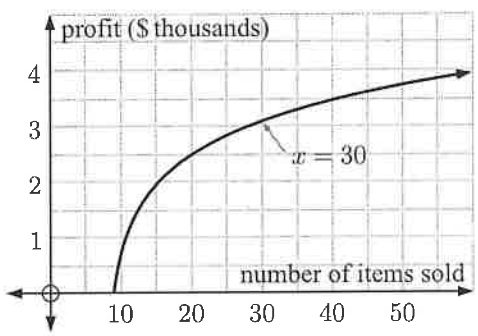
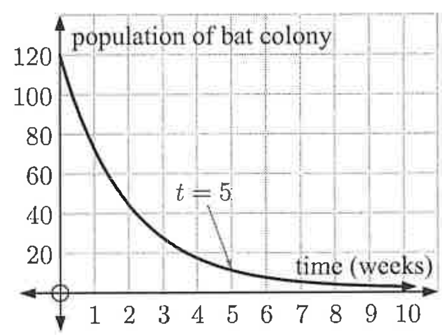


d



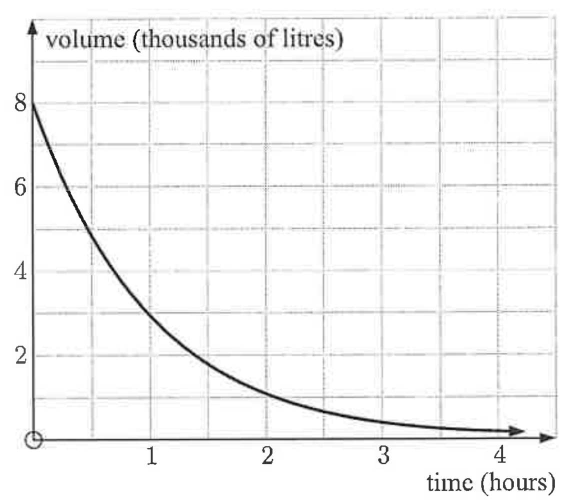
2 Water is leaking from a tank. The volume of water left in the tank over time is given in the graph alongside.

a How much water was in the tank originally?

b How much water was in the tank after 1 hour?

c How quickly was the tank losing water initially?

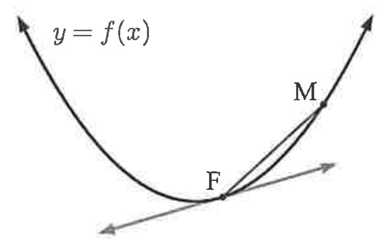
d How quickly was the tank losing water after 1 hour?



FINDING THE GRADIENT OF A TANGENT ALGEBRAICALLY

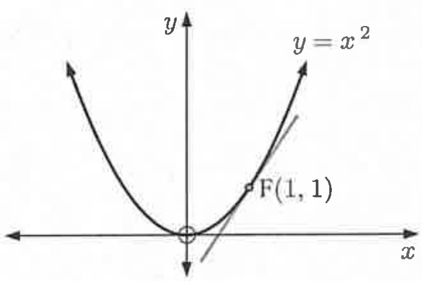
In the previous exercise we found the instantaneous rate of change at a given point by drawing the tangent at that point and finding its gradient. The problem with this method is that it is difficult to draw an accurate tangent by hand, and the result will vary from one person to the next. So, we need a better method for finding the gradient of a tangent.

Consider the curve $y = f(x)$. We wish to find the gradient of the tangent to the curve at the fixed point F. To do this we add a moving point M to the curve, and observe what happens to the gradient of the chord FM as M is moved closer and closer to F.



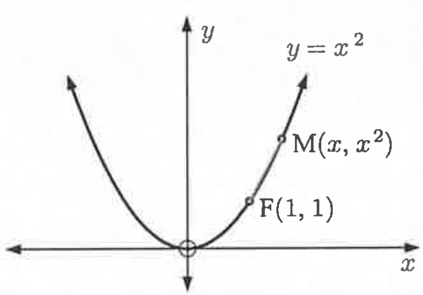
INVESTIGATION 3 THE GRADIENT OF A TANGENT

Given the curve $y = x^2$, we wish to find the gradient of the tangent at the point F(1, 1).



What to do:

1



Suppose M lies on $y = x^2$ and M has coordinates (x, x^2).

Copy and complete the table below:

x	Point M	gradient of FM
5	(5, 25)	$\frac{25-1}{5-1} = 6$
3		
2		
1.5		
1.1		
1.01		
1.001		

2 Comment on the gradient of FM as x gets closer to 1.

3 Repeat the process as x gets closer to 1, but from the left of F.

4 Click on the icon to view a demonstration of the process.

5 What do you suspect is the gradient of the tangent at F?

DEMO

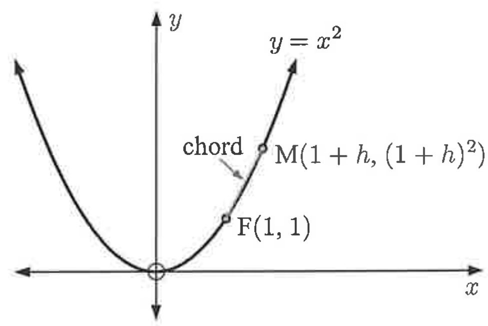
Fortunately we do not have to use a graph and table of values each time we wish to find the gradient of a tangent. Instead we can use an algebraic approach.

THE ALGEBRAIC METHOD

To illustrate the algebraic method we will once again consider the curve $y = x^2$ and the tangent at F(1, 1).

Let the moving point M have x-coordinate $1 + h$, where $h \neq 0$.

So, M is at $(1+h, (1+h)^2)$.



The gradient of chord FM is

$$\frac{y\text{-step}}{x\text{-step}} = \frac{(1+h)^2 - 1}{1+h-1}$$
$$= \frac{\not{1} + 2h + h^2 - \not{1}}{h}$$
$$= \frac{2h + h^2}{h}$$
$$= \frac{\not{h}(2+h)}{\not{h}}$$
$$= 2 + h \qquad \{\text{as } h \neq 0\}$$

Now as M approaches F, h approaches 0, and so $2 + h$ approaches 2.

Since the gradient of the chord FM gets closer and closer to 2 as M approaches F, we conclude that the tangent at (1, 1) has gradient 2.

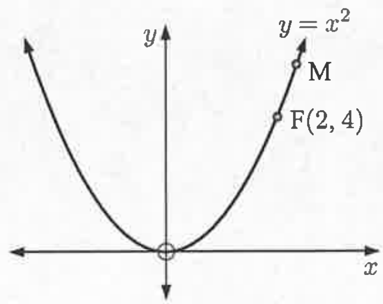
Therefore, the tangent at (1, 1) has gradient 2.

Example 3 **Self Tutor**

Use the algebraic method to find the gradient of the tangent to $y = x^2$ at the point where $x = 2$.

When $x = 2$, $y = 2^2 = 4$, so the fixed point F is (2, 4).

Let $\mathrm{M}(2+h, (2+h)^2)$ be a point on $y = x^2$ which is close to F.



The gradient of FM $= \dfrac{y\text{-step}}{x\text{-step}}$

$$= \frac{(2+h)^2 - 4}{2+h-2}$$
$$= \frac{\not{4} + 4h + h^2 - \not{4}}{h} \qquad \{\text{using } (a+b)^2 = a^2 + 2ab + b^2\}$$
$$= \frac{\not{h}(4+h)}{\not{h}}$$
$$= 4 + h \qquad \{\text{as } h \neq 0\}$$

Now as M approaches F, h approaches 0, and $4 + h$ approaches 4.

So, the tangent at (2, 4) has gradient 4.

EXERCISE 20B.2

1 Use the algebraic method to find the gradient of the tangent to:

a $y = x^2$ at the point where $x = 3$

b $y = 3x^2$ at the point where $x = 2$

c $y = -x^2 - 2x$ at the point where $x = 0$

d $y = x^2 + 3x$ at the point where $x = 1$

e $y = 2x - x^2$ at the point where $x = 3$.

2 **a** Using $(x+h)^3 = (x+h)^2(x+h)$, show that $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$.

b Find $(1+h)^3$ in expanded form using **a**.

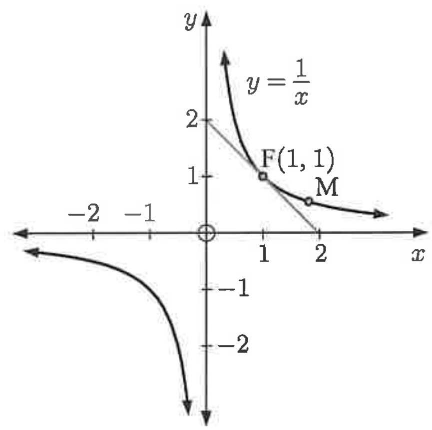
c Consider finding the gradient of the tangent to $y = x^3$ at the point F(1, 1). If the x-coordinate of a moving point M, which is close to F, has value $1 + h$, state the coordinates of M.

d Find the gradient of the chord FM in simplest form.

e What is the gradient of the tangent to $y = x^3$ at the point (1, 1)?

3 Find the gradient of the tangent to $y = x^3$ at the point (2, 8).

4 Consider the graph of $y = \dfrac{1}{x}$ alongside.



a Show that $\dfrac{1}{x+h} - \dfrac{1}{x} = \dfrac{-h}{x(x+h)}$.

b Suppose M has x-coordinate $1 + h$.

i State the y-coordinate of M.

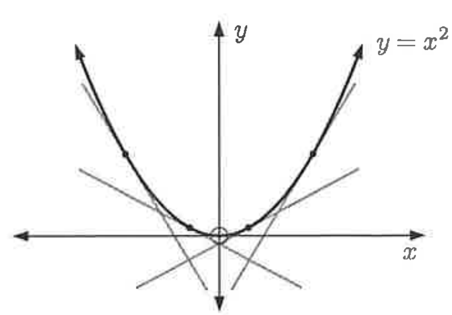
ii Find the gradient of FM in terms of h.

c Find the gradient of the tangent to $y = \dfrac{1}{x}$ at the point where $x = 1$.

d Find the gradient of the tangent to $y = \dfrac{1}{x}$ at the point where $x = 3$.

C THE DERIVATIVE FUNCTION

For a non-linear function, the gradient of the tangent changes as we move along the graph.



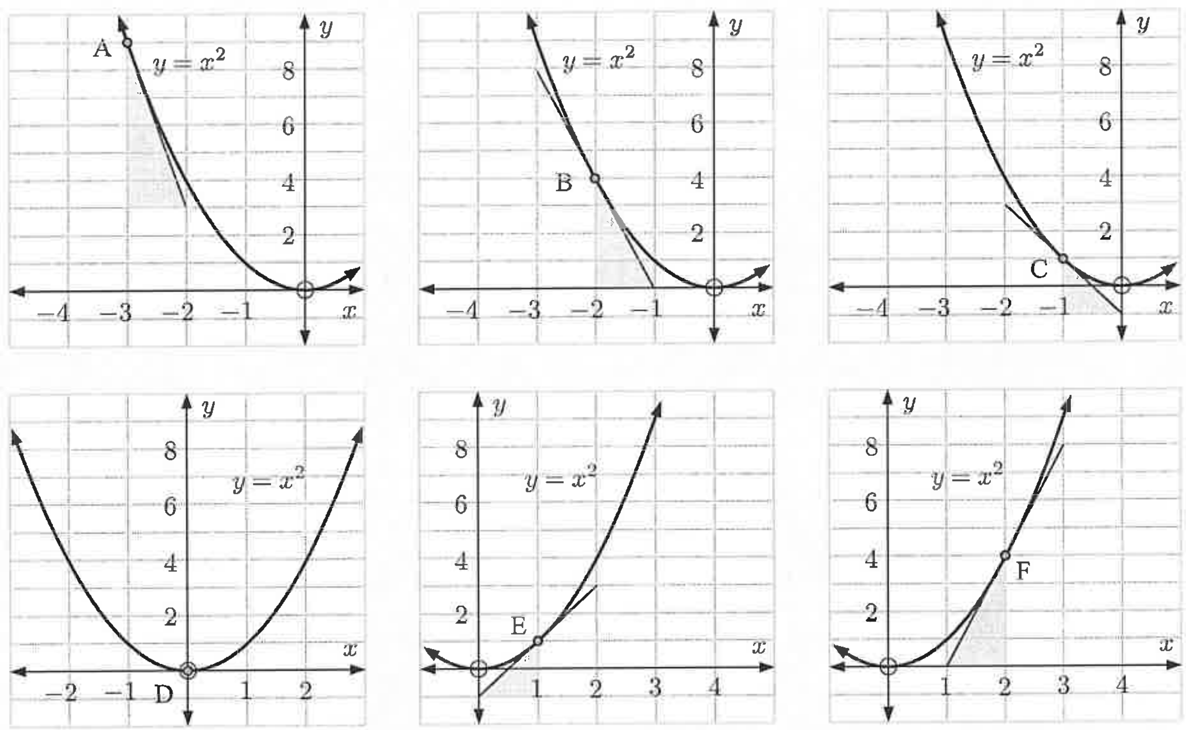
We can hence describe a **gradient function** which, for any given value of x, gives the gradient of the tangent at that point. We call this gradient function the **derived function** or **derivative function** of the curve.

If we are given y in terms of x, we represent the derivative function by $\dfrac{dy}{dx}$. We say this as 'dee y by dee x'.

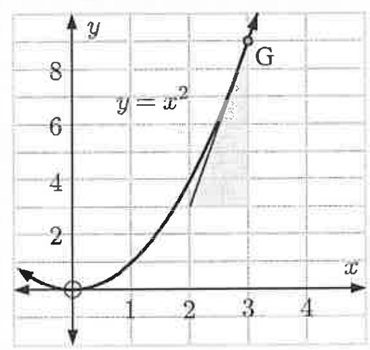
If we are given the function $f(x)$, we represent the derivative function by $f'(x)$. We say this as 'f dashed x'.

INVESTIGATION 4 THE DERIVATIVE OF $y = x^2$

The graphs below show $y = x^2$ with tangents drawn at the points where $x = -3, -2, ..., 3$.



What to do:



1 Use the shaded triangles to find the gradients of the tangents to $y = x^2$ at the seven different points. Hence complete the following table:

x-coordinate	-3						
gradient of tangent							

2 Use your table to help complete:
"the gradient of the tangent to $y = x^2$ at (x, y) is $m = ...$"

DEMO

3 Click on the icon to check the validity of your statement in **2**.
Click on the bar at the top to drag the point of contact of the tangent along the curve.

You should have found that the gradient of the tangent to $y = x^2$ at the point (x, y) is given by $2x$.

So, $y = x^2$ has the derivative function $\frac{dy}{dx} = 2x$.

Alternatively, if $f(x) = x^2$ then $f'(x) = 2x$.

INVESTIGATION 5 THE DERIVATIVE OF $y = x^n$

DERIVATIVE DETERMINER

In this investigation we seek derivative functions for other functions of the form $y = x^n$ where $n \in \mathbb{Z}$.

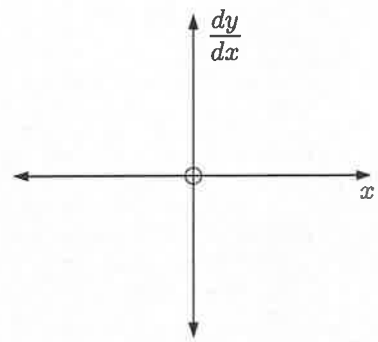
What to do:

1 Click on the icon to run the derivative determiner software.

2 Choose the function $y = x$. By sliding the point along the graph, we can observe the changing gradient of the tangent.

a Use the software to complete the table:

x	-3	-2	-1	0	1	2	3
$\frac{dy}{dx}$							



b Graph $\frac{dy}{dx}$ against x.

Hence predict a formula for the derivative function $\frac{dy}{dx}$.

3 Repeat **2** for the functions:

a $y = x^3$ **b** $y = x^4$ **c** $y = \frac{1}{x} = x^{-1}$ **d** $y = \frac{1}{x^2} = x^{-2}$

Hint: When $x = 0$, the derivatives of both $y = x^{-1}$ and $y = x^{-2}$ are undefined.

4 Use your results to complete this table of derivative functions:

Function	*Derivative function*
x	
x^2	$2x$
x^3	
x^4	
x^{-1}	
x^{-2}	

5 For $y = x^n$, $n \in \mathbb{Z}$, predict the form of $\frac{dy}{dx}$.

You should have discovered that:

$$\text{If } y = x^n \text{ then } \frac{dy}{dx} = nx^{n-1}.$$

DISCUSSION

Does the rule "if $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$" work when $n = 0$?

Example 4 **Self Tutor**

Find the gradient function for:

a $y = x^8$ **b** $f(x) = \dfrac{1}{x^3}$

a $y = x^8$

$\therefore \dfrac{dy}{dx} = 8x^7$

b $f(x) = \dfrac{1}{x^3} = x^{-3}$

$\therefore f'(x) = -3x^{-4}$

Remember that $\dfrac{1}{x^n} = x^{-n}$.

Once we have found the derivative function, we can substitute a value of x to find the gradient of the tangent at that point.

Example 5 **Self Tutor**

Consider $f(x) = \dfrac{1}{x^2}$.

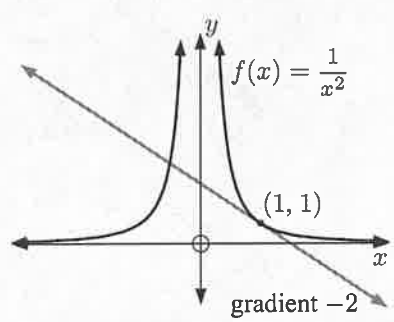
a Find $f'(x)$. **b** Find and interpret $f'(1)$.

a $f(x) = \dfrac{1}{x^2} = x^{-2}$

$\therefore f'(x) = -2x^{-3} = -\dfrac{2}{x^3}$

b $f'(1) = -\dfrac{2}{1^3} = -2$

The tangent to $f(x) = \dfrac{1}{x^2}$ at the point where $x = 1$, has gradient -2.



$f'(1)$ gives the gradient of the tangent to $y = f(x)$ at the point where $x = 1$.

EXERCISE 20C

1 Find the gradient function $\dfrac{dy}{dx}$ for:

a $y = x^6$ **b** $y = \dfrac{1}{x^5}$ **c** $y = x^9$ **d** $y = \dfrac{1}{x^7}$

2 For $f(x) = x^5$, find:

a $f(2)$ **b** $f'(2)$ **c** $f(-1)$ **d** $f'(-1)$

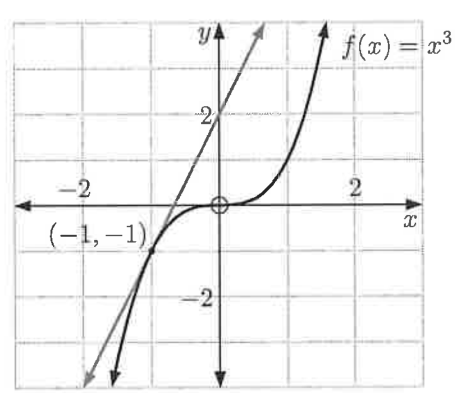
3 Consider $f(x) = \dfrac{1}{x^4}$.

a Find $f'(x)$. **b** Find and interpret $f'(1)$.

4 The graph of $f(x) = x^3$ is shown alongside, and its tangent at the point $(-1, -1)$.

a Use the graph to find the gradient of the tangent.

b Check your answer by finding $f'(-1)$.



D RULES OF DIFFERENTIATION

Differentiation is the process of finding a derivative or gradient function.

There are a number of rules associated with differentiation, which can be used to differentiate more complicated functions.

INVESTIGATION 6 RULES OF DIFFERENTIATION

In this investigation we attempt to differentiate functions of the form ax^n where a is a constant, and functions which are a sum or difference of terms of the form ax^n.

DERIVATIVE DETERMINER

What to do:

1 Use the software to find the derivatives of:

a x^2 **b** $4x^2$ **c** x^3 **d** $2x^3$

Hence copy and complete: "If $f(x) = ax^n$ then $f'(x) = ...$"

2 Use the software to find the derivatives of:

a $f(x) = x^2 + 3x$ **b** $f(x) = x^3 - 2x^2$

Hence copy and complete: "If $f(x) = u(x) + v(x)$ then $f'(x) = ...$"

We have now determined the following rules for differentiating:

Function	$f(x)$	$f'(x)$
a constant	a	0
x^n	x^n	nx^{n-1}
a constant multiple of x^n	ax^n	anx^{n-1}
multiple terms	$u(x) + v(x)$	$u'(x) + v'(x)$

Using the rules we have now developed we can differentiate sums of powers of x.

For example, if $f(x) = 3x^4 + 2x^3 - 5x^2 + 7x + 6$ then

$$\begin{aligned} f'(x) &= 3(4x^3) + 2(3x^2) - 5(2x) + 7(1) + 0 \\ &= 12x^3 + 6x^2 - 10x + 7 \end{aligned}$$

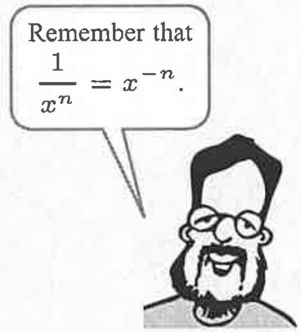
Example 6 **Self Tutor**

Find $f'(x)$ for $f(x)$ equal to:

a $5x^3 + 6x^2 - 3x + 2$ **b** $7x - \frac{4}{x} + \frac{3}{x^3}$ **c** $\frac{x^2 + 4x - 5}{x}$

a $$\begin{aligned} f(x) &= 5x^3 + 6x^2 - 3x + 2 \\ \therefore\ f'(x) &= 5(3x^2) + 6(2x) - 3(1) \\ &= 15x^2 + 12x - 3 \end{aligned}$$

b $$f(x) = 7x - \frac{4}{x} + \frac{3}{x^3}$$
$$= 7x - 4x^{-1} + 3x^{-3}$$
$$\therefore \ f'(x) = 7(1) - 4(-1x^{-2}) + 3(-3x^{-4})$$
$$= 7 + 4x^{-2} - 9x^{-4}$$
$$= 7 + \frac{4}{x^2} - \frac{9}{x^4}$$



c $$f(x) = \frac{x^2 + 4x - 5}{x}$$
$$= \frac{x^2}{x} + 4 - \frac{5}{x}$$
$$= x + 4 - 5x^{-1}$$
$$\therefore \ f'(x) = 1 + 5x^{-2}$$

EXERCISE 20D

1 Find $f'(x)$ given that $f(x)$ is:

a x^3 **b** $2x^3$ **c** $7x^2$ **d** $x^2 + x$
e $4 - 2x^2$ **f** $x^2 + 3x - 5$ **g** $5x^4 - 6x^2$ **h** $x^3 + 3x^2 + 4x - 1$
i $3 - 6x^{-1}$ **j** $\frac{2x - 3}{x^2}$ **k** $\frac{x^3 + 5}{x}$ **l** $\frac{x^3 + x - 3}{x}$

2 Suppose $f(x) = 4x^3 - x$. Find:

a $f'(x)$ **b** $f'(2)$ **c** $f'(0)$

3 Suppose $g(x) = \frac{x^2 + 1}{x}$. Find:

a $g'(x)$ **b** $g'(3)$ **c** $g'(-2)$

Example 7 **Self Tutor**

Find the gradient function of $f(x) = x^2 - \frac{4}{x}$.

Hence find the gradient of the tangent to the function at the point where $x = 2$.

$$f(x) = x^2 - \frac{4}{x}$$
$$= x^2 - 4x^{-1}$$
$$\therefore \ f'(x) = 2x - 4(-1x^{-2})$$
$$= 2x + 4x^{-2}$$
$$= 2x + \frac{4}{x^2}$$

Now $f'(2) = 2(2) + \frac{4}{2^2} = 5$.

So, the tangent has gradient $= 5$.

4 Find the gradient of the tangent to:

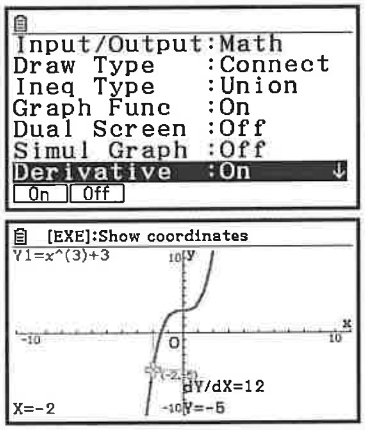
a $y = x^2$ at $x = 2$
b $y = \frac{8}{x^2}$ at $x = 9$
c $y = 2x^2 - 3x + 7$ at $x = -1$
d $y = 2x - 5x^{-1}$ at $x = 2$
e $y = \frac{x^2 - 4}{x^2}$ at $x = 4$
f $y = \frac{x^3 - 4x - 8}{x^2}$ at $x = -1$

Example 8 **Self Tutor**

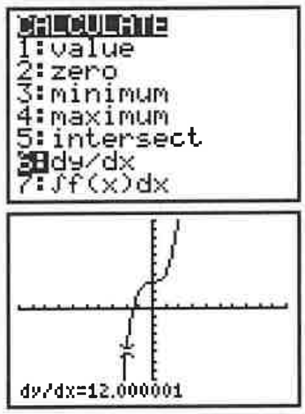
Use technology to find the gradient of the tangent to $y = x^3 + 3$ at the point where $x = -2$.

GRAPHICS CALCULATOR INSTRUCTIONS

Casio fx-CG20



TI-84 Plus



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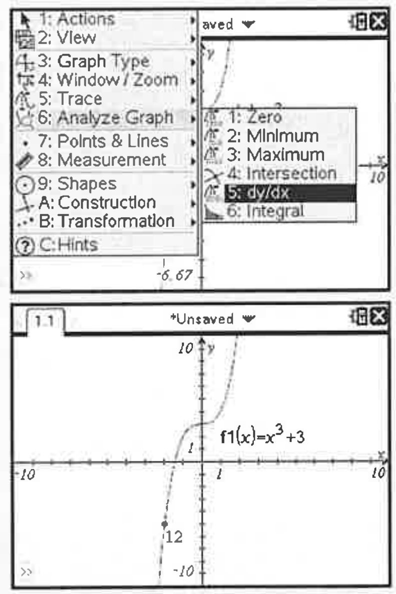


So, the gradient of the tangent is 12.

5 Consider the function $f(x) = (3x + 1)^2$.

a Expand the brackets of $(3x + 1)^2$.

b Hence find $f'(x)$.

c Hence find the gradient of the tangent to $y = f(x)$ at the point where $x = -2$.

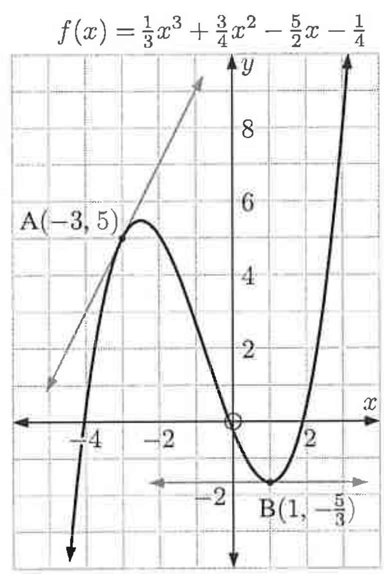
Check your answer to **c** using technology.

6 The graph of $f(x) = \frac{1}{3}x^3 + \frac{3}{4}x^2 - \frac{5}{2}x - \frac{1}{4}$ is shown alongside. The tangents at A$(-3, 5)$ and B$(1, -\frac{5}{3})$ are also given.

a Use the graph to find the gradients of the tangents at A and B.

b Calculate the gradients in **a** exactly by finding $f'(-3)$ and $f'(1)$.

Check your answers using technology.



Example 9 — Self Tutor

If $y = 3x^2 - 4x$, find $\frac{dy}{dx}$ and interpret its meaning.

As $y = 3x^2 - 4x$, $\frac{dy}{dx} = 6x - 4$.

$\frac{dy}{dx}$ is:
- the gradient function or derivative function of $y = 3x^2 - 4x$, from which the gradient at any point can be found
- the instantaneous rate of change in y as x changes.

7 If $y = 4x - \frac{3}{x}$, find $\frac{dy}{dx}$ and interpret its meaning.

8 The position of a car moving along a straight road is given by $S = 2t^2 + 4t$ metres, where t is the time in seconds.

Find $\frac{dS}{dt}$ and interpret its meaning.

9 The cost of producing and selling x toasters each week is given by $C = 1785 + 3x + 0.002x^2$ dollars.

Find $\frac{dC}{dx}$ and interpret its meaning.

Example 10 — Self Tutor

At what point on the graph of $y = 2x^2 + 5x - 3$ does the tangent have gradient 13?

Since $y = 2x^2 + 5x - 3$, $\frac{dy}{dx} = 4x + 5$

$\therefore$ the tangent has gradient 13 when $4x + 5 = 13$
$\therefore\ 4x = 8$
$\therefore\ x = 2$

When $x = 2$, $y = 2(2)^2 + 5(2) - 3 = 15$

So, the tangent has gradient 13 at the point (2, 15).

10 At what point on the graph of $y = x^2 - 4x + 7$ does the tangent have gradient 2? Draw a diagram to illustrate your answer.

11 Find the coordinates of the point(s) on:

a $y = x^2 + 5x + 1$ where the tangent has gradient 3

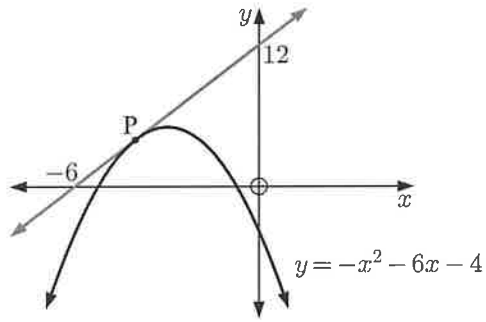
b $y = 3x^2 + 11x + 5$ where the tangent has gradient -7

c $f(x) = 2x^{-2} + x$ where the tangent has gradient $\frac{1}{2}$

d $f(x) = 3x^3 - 5x + 2$ where the tangent has gradient 4

e $f(x) = ax^2 + bx + c$ where the tangent is horizontal.

12 Find the coordinates of P on the graph shown.



Example 11 **Self Tutor**

The tangent to $f(x) = 2x^2 - ax + b$ at the point (2, 7) has a gradient of 3. Find a and b.

Since $f(x) = 2x^2 - ax + b$, $f'(x) = 4x - a$

Now, $f'(2) = 3$, so $4(2) - a = 3$
$\therefore\ a = 5$

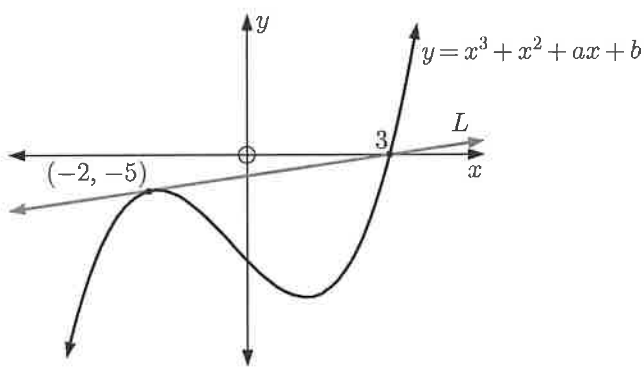
Also, $f(2) = 7$, so $2(2)^2 - 5(2) + b = 7$
$\therefore\ b = 9$

13 The tangent to $f(x) = x^3 + ax + 5$ at the point where $x = 1$, has a gradient of 10. Find a.

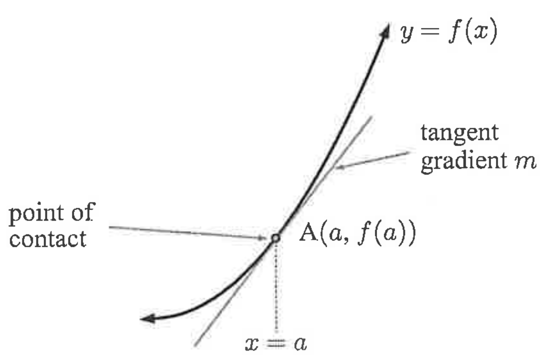
14 The tangent to $f(x) = -3x^2 + ax + b$ at the point $(-3, 8)$ has a gradient of 9. Find a and b.

15 The tangent to $f(x) = 2x^2 + a + \frac{b}{x}$ at the point (1, 11) has a gradient of -2. Find a and b.

16 **a** Find the gradient of the tangent line L.
b Hence, find a and b.



E EQUATIONS OF TANGENTS



Consider a curve $y = f(x)$.

If the point A has x-coordinate a, then its y-coordinate is $f(a)$, and the gradient of the tangent at A is $f'(a)$.

The equation of the tangent is

$$\frac{y - f(a)}{x - a} = f'(a) \quad \{\text{equating gradients}\}$$

or $y - f(a) = f'(a)(x - a)$.

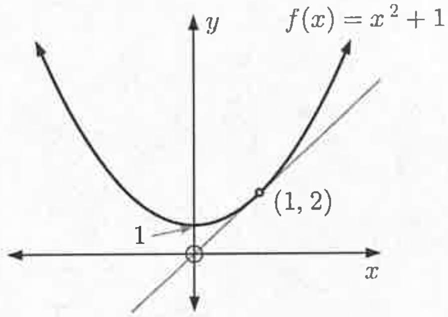
The equation of the tangent at the point $A(a, b)$ is

$$\frac{y-b}{x-a} = f'(a) \quad \text{or} \quad y - b = f'(a)(x-a).$$

The equation can then be rearranged into gradient-intercept form.

Example 12 — Self Tutor

Find the equation of the tangent to $f(x) = x^2 + 1$ at the point where $x = 1$.



Since $f(1) = 1^2 + 1 = 2$,
the point of contact is $(1, 2)$.

Now $f'(x) = 2x$, so $f'(1) = 2$

$\therefore$ the tangent has equation

$$\frac{y-2}{x-1} = 2$$

which is $y - 2 = 2x - 2$

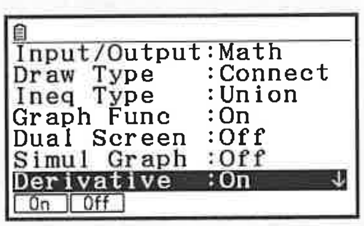
or $y = 2x$.

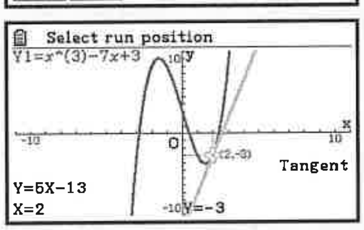
Example 13 — Self Tutor

Use technology to find the equation of the tangent to $y = x^3 - 7x + 3$ at the point where $x = 2$.

GRAPHICS CALCULATOR INSTRUCTIONS

Casio fx-CG20





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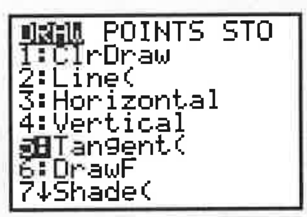

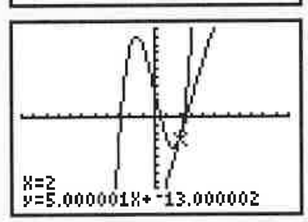

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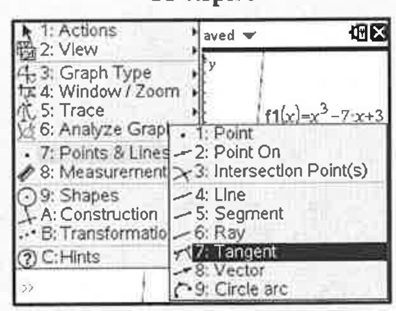
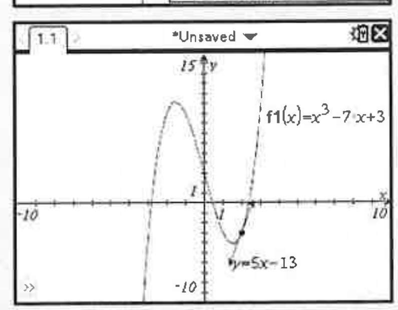




So, the tangent has equation $y = 5x - 13$.

EXERCISE 20E

1 Find the equation of the tangent to:

a $y = x^2$ at $x = 4$

b $y = x^3$ at $x = -2$

c $y = 3x^{-1}$ at $x = -1$

d $y = \dfrac{4}{x^3}$ at $x = 2$

e $y = x^2 + 5x - 4$ at $x = 1$

f $y = 2x^2 + 5x + 3$ at $x = -2$

g $y = x^3 + 2x$ at $x = 0$

h $y = x^2 + x^{-1}$ at $x = 0$

i $y = x + 2x^{-1}$ at $x = 2$

j $y = \dfrac{x^2 + 4}{x}$ at $x = -1$

Check your answers using technology.

Example 14 — Self Tutor

Consider the curve $y = x^3 - 4x^2 - 6x + 8$.

a Find the equation of the tangent to this curve at the point where $x = 0$.

b At what point does this tangent meet the curve again?

a When $x = 0$, $y = 8$. So, the point of contact is $(0, 8)$.

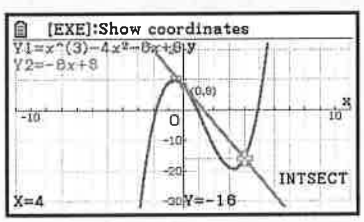
$\dfrac{dy}{dx} = 3x^2 - 8x - 6$, so when $x = 0$, $\dfrac{dy}{dx} = -6$.

$\therefore$ the tangent has equation $\dfrac{y - 8}{x - 0} = -6$

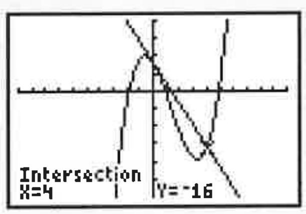
$\therefore$ $y = -6x + 8$

b We use technology to find where the tangent meets the curve again:

Casio fx-CG20



TI-84 Plus



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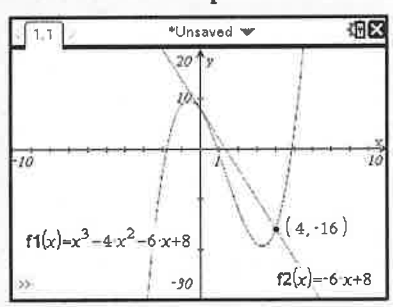


The tangent meets the curve again at $(4, -16)$.

2 For each of the following curves:

- **i** find the equation of the tangent at the given point
- **ii** find the point at which this tangent meets the curve again.

a $f(x) = 2x^3 - 5x + 1$ at $x = -1$

b $y = x^2 + \dfrac{3}{x} + 2$ at $x = 3$

c $f(x) = x^3 + 5$ at $x = 1.5$

d $y = x^3 + \dfrac{1}{x}$ at $x = -1$

e $f(x) = 3x^3 + 2x^2 - x + 2$ at $x = 0.5$

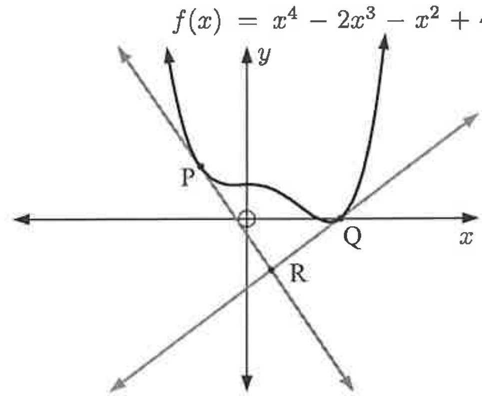
3 **a** Find the point where the tangent to $y = 2x^3 + 3x^2 - x + 4$ at $x = -1$ meets the x-axis.

b Find the point where the tangent to $y = x^3 + 5$ at $(-2, -3)$ meets the line $y = 2$.

c Find the point where the tangent to $y = \frac{2}{x} + 1$ at $(-2, 0)$ meets the line $y = 2x - 3$.

d Find the point where the tangent to $y = 3x^3 - 2x + 1$ at $x = 1$ meets the y-axis.

4 The graph of $f(x) = x^4 - 2x^3 - x^2 + 4$ is shown alongside. The tangents at P$(-1, 6)$ and Q$(2, 0)$ intersect at R.
Find the coordinates of R.



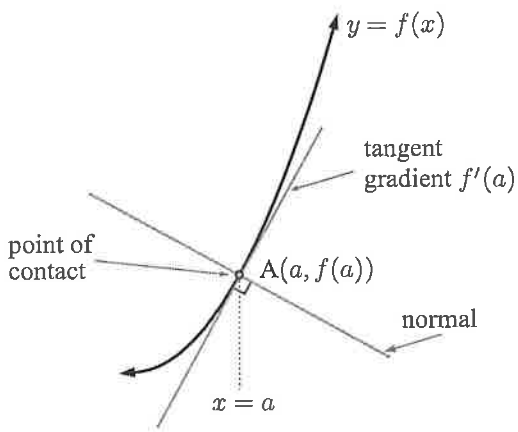
5 Consider the function $f(x) = x^4 - 2x^2 + 2x + 3$.

a Find the equation of the line which is the tangent to the curve at the point where $x = 1$.

b Find the point at which this line meets the curve again.

c Show that the line is the tangent to the curve at this point also.

F NORMALS TO CURVES



A **normal** to a curve is a line which is perpendicular to the tangent at the point of contact.

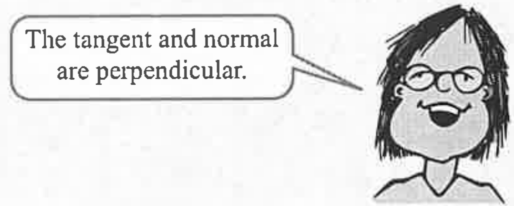
The gradients of perpendicular lines are negative reciprocals of each other, so:

The gradient of the **normal** at $x = a$ is $-\dfrac{1}{f'(a)}$.

Example 15 **Self Tutor**

Find the equation of the normal to $f(x) = x^2 - 4x + 3$ at the point where $x = 4$.

Since $f(4) = (4)^2 - 4(4) + 3 = 3$,
the point of contact is $(4, 3)$.



Now $f'(x) = 2x - 4$,
so $f'(4) = 2(4) - 4$
$= 4$

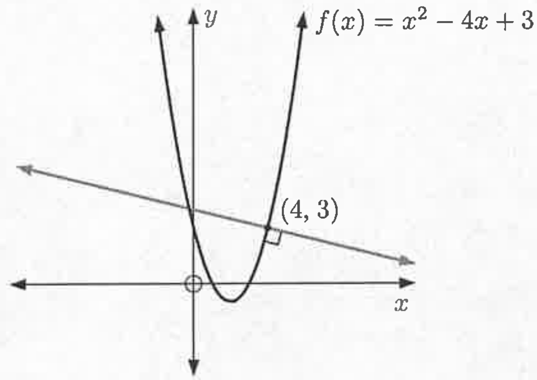
So, the normal at $(4, 3)$ has gradient $-\frac{1}{4}$.

$\therefore$ the normal has equation

$$\frac{y-3}{x-4} = -\frac{1}{4}$$

which is $y - 3 = -\frac{1}{4}x + 1$

or $y = -\frac{1}{4}x + 4$



Example 16 **Self Tutor**

Find the coordinates of the point where the normal to $y = x^2 - 3$ at $(1, -2)$ meets the curve again.

$\frac{dy}{dx} = 2x$, so when $x = 1$, $\frac{dy}{dx} = 2$

So, the normal at $(1, -2)$ has gradient $-\frac{1}{2}$.

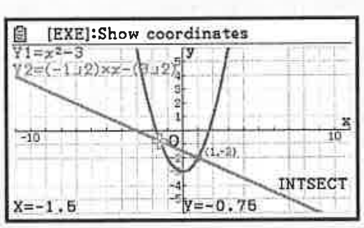
$\therefore$ the normal has equation $\frac{y-(-2)}{x-1} = -\frac{1}{2}$

$\therefore\ y + 2 = -\frac{1}{2}x + \frac{1}{2}$

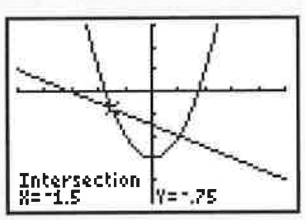
$\therefore\ y = -\frac{1}{2}x - \frac{3}{2}$

We use technology to find where the normal meets the curve again:

Casio fx-CG20



TI-84 Plus



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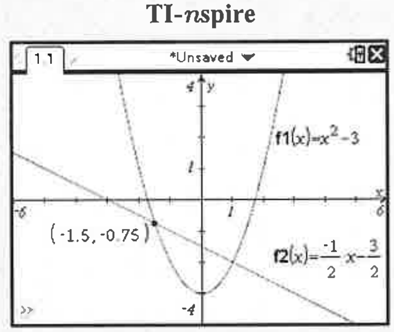


The normal meets the curve again at $(-1.5, -0.75)$.

EXERCISE 20F

1 Find the equation of the normal to:

a $y = x^2$ at the point $(3, 9)$

b $y = x^3 - 5x + 2$ at $x = -2$

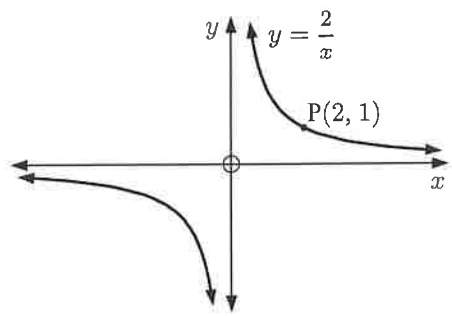
c $y = \frac{1}{x} + 2$ at the point $(-1, 1)$

d $y = 2x^3 - 3x + 1$ at $x = 1$

e $y = x^2 - 3x + 2$ at $x = 3$

f $y = 3x + \frac{1}{x} - 4$ at $x = 1$.

2 Consider the graph of $y = \frac{2}{x}$ alongside.



a Find the equation of:

i the tangent at P

ii the normal at P.

b Sketch the graph of $y = \frac{2}{x}$, including the tangent and normal at P.

3 For each of the following curves, find the coordinates of the point where the normal to the curve at the given point, meets the curve again.

a $y = x^2$ at $(2, 4)$

b $y = \frac{1}{x} + 2$ at $(-1, 1)$

c $y = x^3$ at $x = -1$

d $y = x^3 - 12x + 2$ at $(3, -7)$.

4 **a** Find where the normal to $y = x^3 - 12x + 2$ at $x = -2$ meets the x-axis.

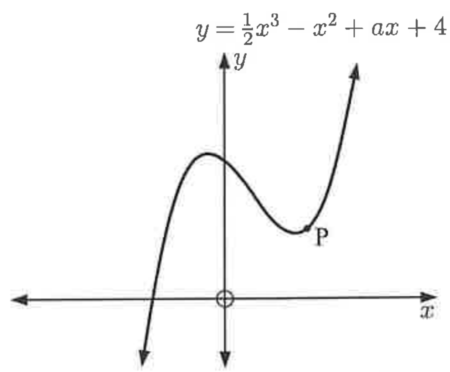
b Find where the normal to $y = x^3$ at $(-1, -1)$ meets the line $y = 3$.

c Find where the normal to $y = \frac{1}{x} - 3$ at $(-1, -4)$ meets the line $y = -2x + 1$.

d Find where the normal to $y = 2x^3 - 3x + 1$ at $x = 1$ meets the y-axis.

5 The normal to $y = 5 - \frac{a}{x}$ at the point where $x = -2$, has a gradient of 1. Find a.

6 In the graph alongside, P is the point with x-coordinate 2.



a The tangent at P has a gradient of 1. Find:

i a

ii the coordinates of P.

b Find the equation of the normal at P.

c Find the coordinates of the point Q where the normal at P meets the curve again.

d Find the equation of the tangent at Q. Comment on your answer.

THEORY OF KNOWLEDGE

The Greek philosopher Zeno of Elea lived in what is now southern Italy, in the 5th century BC. He is most famous for his paradoxes, which were recorded in Aristotle's work *Physics*.

The arrow paradox

"*If everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless.*"

This argument says that if we fix an instant in time, an arrow appears motionless. Consequently, how is it that the arrow actually moves?

The dichotomy paradox

"*That which is in locomotion must arrive at the half-way stage before it arrives at the goal.*"

If an object is to move a fixed distance then it must travel half that distance. Before it can travel a half the distance, it must travel a half *that* distance. With this process continuing indefinitely, motion is impossible.

Achilles and the tortoise

"*In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead.*"

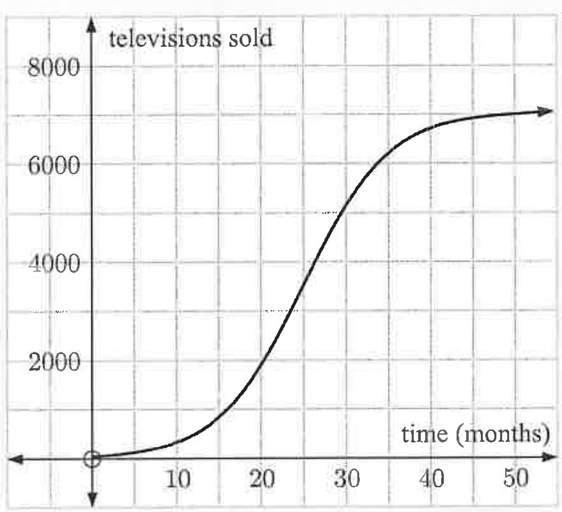
According to this principle, the athlete Achilles will never be able to catch the slow tortoise!

1. A paradox is a logical argument that leads to a contradiction or a situation which defies logic or reason. Can a paradox be the truth?
2. Are Zeno's paradoxes really paradoxes?
3. Are the three paradoxes essentially the same?
4. We know from experience that things *do* move, and that Achilles *would* catch the tortoise. Does that mean that logic has failed?
5. What do Zeno's paradoxes have to do with rates of change?

REVIEW SET 20A

1. The total number of televisions sold over many months is shown on the graph alongside.
   Estimate the rate of sales:
   a from 40 to 50 months
   b from 0 to 50 months
   c at 20 months.

PRINTABLE GRAPH



2 Use the rules of differentiation to find $f'(x)$ for $f(x)$ equal to:

a $7x^3$ **b** $3x^2 - x^3$ **c** $(2x-3)^2$ **d** $\dfrac{7x^3 + 2x^4}{x^2}$

3 Consider $f(x) = x^4 - 3x - 1$. Find: **a** $f'(x)$ **b** $f'(2)$ **c** $f'(0)$.

4 Find the equation of the tangent to $y = -2x^2$ at the point where $x = -1$.

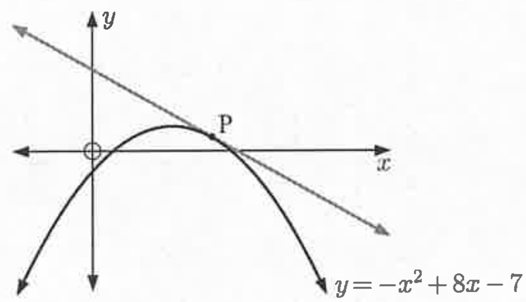
5 Consider the function $f(x) = -2x^2 + 5x + 3$. Find:

a the average rate of change from $x = 2$ to $x = 4$

b the instantaneous rate of change at $x = 2$.

6 Find the equation of the normal to $y = x^3 + 3x - 2$ at the point where $x = 2$.

7 The tangent shown has gradient -4.
Find the coordinates of P.



8 The tangent to $y = ax^3 - 3x + 3$ at the point where $x = 2$, has a gradient of 21. Find a.

REVIEW SET 20B

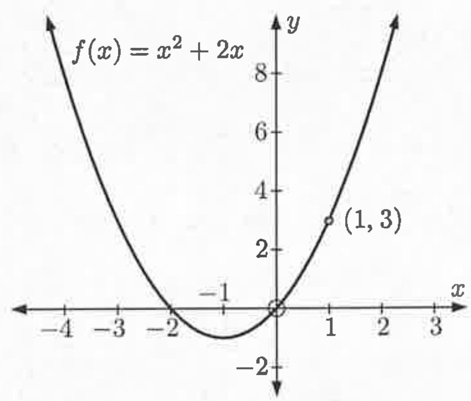
1 Consider the function $f(x) = x^2 + 2x$, which has the graph shown.

a Find the gradient of the line which passes through (1, 3) and the point on $f(x)$ with x-coordinate:

i 2 **ii** 1.5 **iii** 1.1

b Find $f'(x)$.

c Find the gradient of the tangent to $f(x)$ at (1, 3). Compare this with your answers to **a**.



2 Find $\dfrac{dy}{dx}$ for:

a $y = 3x^2 - x^4$ **b** $y = \dfrac{x^3 - x}{x^2}$ **c** $y = 2x + x^{-1} - 3x^{-2}$

3 Find the equation of the tangent to $y = x^3 - 3x + 5$ at the point where $x = 2$.

4 Find all points on the curve $y = 2x + x^{-1}$ where the tangent is horizontal.

5 If $f(x) = 7 + x - 3x^2$, find: **a** $f(3)$ **b** $f'(3)$.

6 Find the coordinates of the point where the normal to $y = x^2 - 7x - 44$ at $x = -3$ meets the curve again.

7 The tangent to $f(x) = a - \frac{b}{x^2}$ at $(-1, -1)$ has equation $y = -6x - 7$. Find the values of a and b.

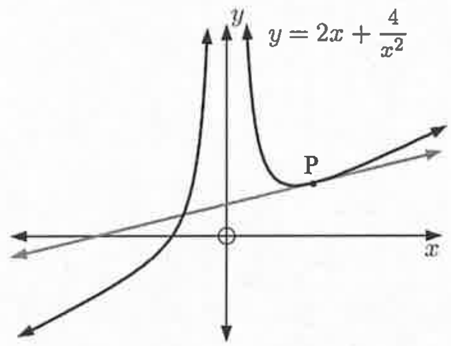
8 The tangent shown has a gradient of 1.

a Find the coordinates of P.

b Find the equation of the tangent.

c Find where the tangent cuts the x-axis.

d Find the equation of the normal at P.



REVIEW SET 20C

1 Use the rules of differentiation to find $f'(x)$ for $f(x)$ equal to:

a $x^4 + 2x^3 + 3x^2 - 5$ **b** $2x^{-3} + x^{-4}$ **c** $\frac{1}{x} - \frac{4}{x^2}$

2 Find the gradient of $f(x)$ at the given point for the following functions:

a $f(x) = x^2 - 3x$ at $x = -1$ **b** $f(x) = -3x^2 + 4$ at $x = 2$

c $f(x) = x + \frac{2}{x}$ at $x = 3$ **d** $f(x) = x^3 - x^2 - x - 2$ at $x = 0$

3 Find the equation of the tangent to $y = \frac{12}{x^2}$ at the point $(1, 12)$.

4 Sand is poured into a bucket for 30 seconds. After t seconds, the weight of sand is $S(t) = 0.3t^3 - 18t^2 + 550t$ grams.
Find and interpret $S'(t)$.

5 Find the equation of the normal to $y = 2 + \frac{1}{x} + 3x$ at the point where $x = 1$.

6 The tangent to $y = x^3 - 2x^2 + ax - b$ at $(2, -1)$ has equation $y = 7x - 15$.
Find the values of a and b.

7 Find the coordinates of the point where the normal to $y = -3x^3 + 5x - 1$ at $x = 0$ meets the curve again.

8 The graph of $f(x) = x^3 - 4x^2 + 4x + 1$ is shown alongside.

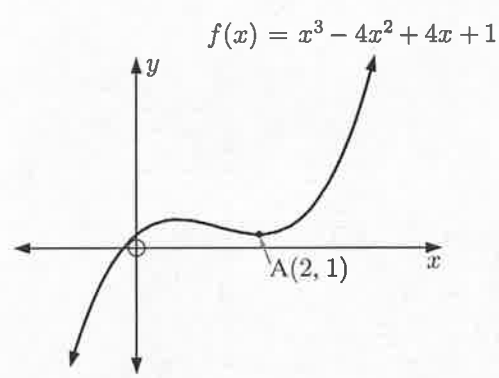
a Find $f'(x)$.

b Find and interpret $f'(1)$.

c The graph has a minimum turning point at A(2, 1).

i Find the gradient of the tangent at A.

ii Find the equation of the tangent at A.



EXERCISE 20A.1

1. a In one minute, Karsten's heart is expected to beat 67 times.
   b 4020 beats per hour
2. a $\approx 0.001\,50$ errors per word
   b ≈ 0.150 errors per 100 words
3. Niko ($12.35)
4. a 0.000 177 mm per km b 1.77 mm per 10 000 km
5. a 89.0 km h^{-1} b 24.7 m s^{-1}

EXERCISE 20A.2

1. a 0.1 m s^{-1} b 0.9 m s^{-1} c 0.5 m s^{-1}
2. a i 3.1 beetles per g ii 4.5 beetles per g
   b The decrease is slow at first, then becomes more rapid from 2 g until 8 g, when the decrease slows down.

EXERCISE 20B.1

1. a 1 m s^{-1} b 3 km h^{-1}
   c $44.40 per item sold d -4.3 bats per week
2. a 8000 L b 3000 L
   c 10 667 L per hour d 3000 L per hour

EXERCISE 20B.2

1. a 6 b 12 c -2 d 5 e -4
2. b $(1+h)^3 = 1 + 3h + 3h^2 + h^3$
   c $\text{M}(1+h,\ 1+3h+3h^2+h^3)$ d h^2+3h+3 e 3
3. 12
4. b i $\dfrac{1}{1+h}$ ii $-\dfrac{1}{1+h}$ c -1 d $-\frac{1}{9}$

EXERCISE 20C

1. a $6x^5$ b $-\dfrac{5}{x^6}$ c $9x^8$ d $-\dfrac{7}{x^8}$
2. a 32 b 80 c -1 d 5
3. a $-\dfrac{4}{x^5}$ b -4 The gradient of the tangent at $x = 1$.
4. a 3

EXERCISE 20D

1. a $f'(x) = 3x^2$ b $f'(x) = 6x^2$
   c $f'(x) = 14x$ d $f'(x) = 2x + 1$
   e $f'(x) = -4x$ f $f'(x) = 2x + 3$
   g $f'(x) = 20x^3 - 12x$ h $f'(x) = 3x^2 + 6x + 4$
   i $f'(x) = 6x^{-2}$ j $f'(x) = -2x^{-2} + 6x^{-3}$
   k $f'(x) = 2x - 5x^{-2}$ l $f'(x) = 2x + 3x^{-2}$

2. a $f'(x) = 12x^2 - 1$ b 47   c -1

3. a $g'(x) = 1 - x^{-2}$ b $\frac{8}{9}$ c $\frac{3}{4}$

4. a 4   b $-\frac{16}{729}$ c -7 d $\frac{13}{4}$ e $\frac{1}{8}$ f -11

5. a $9x^2 + 6x + 1$ b $18x + 6$ c -30

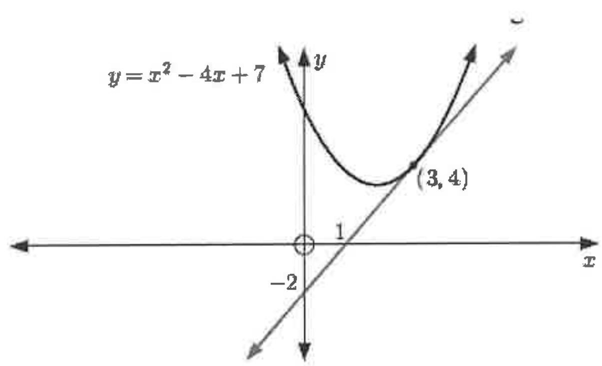
6. a at A, gradient = 2; at B, gradient = 0
   b $f'(x) = x^2 + \frac{3}{2}x - \frac{5}{2}$, $f'(-3) = 2$, $f'(1) = 0$

7. $\dfrac{dy}{dx} = 4 + \dfrac{3}{x^2}$ This is the instantaneous rate of change in y as x changes.

8. $\dfrac{dS}{dt} = 4t + 4$ This gives the speed of the car at time t, in metres per second.

9. $\dfrac{dC}{dx} = 3 + 0.004x$ This is the instantaneous rate of change in production cost as the number of toasters changes.

10.



The tangent has gradient 2 at the point (3, 4).

11. a $(-1, -3)$ b $(-3, -1)$ c $(2, \frac{5}{2})$
    d $(-1, 4)$ or $(1, 0)$ e $\left(-\dfrac{b}{2a}, \dfrac{b^2}{4a} - \dfrac{b^2}{2a} + c\right)$

12. P(−4, 4)   13. $a = 7$ 14. $a = -9$, $b = 8$

15. $a = 3$, $b = 6$ 16. a 1   b $a = -7$, $b = -15$

EXERCISE 20E

1 a $y = 8x - 16$ b $y = 12x + 16$ c $y = -3x - 6$
d $y = -\frac{3}{4}x + 2$ e $y = 7x - 5$ f $y = -3x - 5$
g $y = 2x$ h no tangent exists at $x = 0$
i $y = \frac{1}{2}x + 2$ j $y = -3x - 8$

2 a i $y = x + 5$ ii $(2, 7)$
b i $y = \frac{17}{3}x - 5$ ii $(-\frac{1}{3}, -\frac{62}{9}) \approx (-0.333, -6.89)$
c i $y = 6.75x - 1.75$ ii $(-3, -22)$
d i $y = 2x$ ii $(1, 2)$
e i $y = 3.25x + 0.75$ ii $(-1.67, -4.67)$

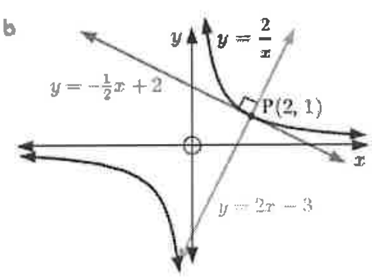
3 a $(5, 0)$ b $(-1.58, 2)$ c $(0.8, -1.4)$ d $(0, -5)$

4 R$(0.5, -6)$ 5 a $y = 2x + 2$ b $(-1, 0)$

EXERCISE 20F

1 a $y = -\frac{1}{6}x + \frac{19}{2}$ b $y = -\frac{1}{7}x + \frac{26}{7}$ c $y = x + 2$
d $y = -\frac{1}{3}x + \frac{1}{3}$ e $y = -\frac{1}{3}x + 3$ f $y = -\frac{1}{2}x + \frac{1}{2}$

2 a i $y = -\frac{1}{2}x + 2$
ii $y = 2x - 3$

b 

3 a $(-2.25, 5.06)$ b $(1, 3)$
c normal does not meet curve again
d $(-3.78, -6.55)$ and $(0.777, -6.85)$

4 a $(-2, 0)$ b $(-13, 3)$ c $(1.33, -1.67)$ d $(0, 0.333)$

5 $a = -4$

6 a i $a = -1$ ii P$(2, 2)$ b $y = -x + 4$ c $(0, 4)$
d $y = -x + 4$. This is the same line as the normal to the curve at P.

REVIEW SET 20A

1 a 33 televisions per month b 140 televisions per month
c 250 televisions per month

2 a $f'(x) = 21x^2$ b $f'(x) = 6x - 3x^2$
c $f'(x) = 8x - 12$ d $f'(x) = 7 + 4x$

3 a $f'(x) = 4x^3 - 3$ b 29 c -3

4 $y = 4x + 2$ 5 a -7 b -3

6 $y = -\frac{1}{15}x + \frac{182}{15}$ or $y \approx 0.0667x + 12.1$

7 P(6, 5) 8 $a = 2$

REVIEW SET 20B

1 a i 5 ii $4\frac{1}{2}$ iii 4.1
b $f'(x) = 2x+2$ c gradient = 4, as $x \to 1$, $f'(x) \to 4$

2 a $\frac{dy}{dx} = 6x - 4x^3$ b $\frac{dy}{dx} = 1 + x^{-2}$
c $\frac{dy}{dx} = 2 - x^{-2} + 6x^{-3}$

3 $y = 9x - 11$ 4 $\left(-\frac{1}{\sqrt{2}}, -2\sqrt{2}\right)$ and $\left(\frac{1}{\sqrt{2}}, 2\sqrt{2}\right)$

5 a -17 b -17 6 $(10.1, -13.0)$ 7 $a = 2$, $b = 3$

8 a P(2, 5) b $y = x + 3$ c $(-3, 0)$ d $y = -x + 7$

REVIEW SET 20C

1 a $f'(x) = 4x^3 + 6x^2 + 6x$ b $f'(x) = -6x^{-4} - 4x^{-5}$
c $f'(x) = -x^{-2} + 8x^{-3}$

2 a -5 b -12 c $\frac{7}{9}$ d -1

3 $y = -24x + 36$

4 $S'(t) = 0.9t^2 - 36t + 550$ g sec^{-1}
This gives the instantaneous rate of change in weight, in grams per second, for a given value of t.

5 $y = -\frac{1}{2}x + \frac{13}{2}$

6 $a = 3$, $b = 7$

7 $(-1.32, -0.737)$ and $(1.32, -1.26)$

8 a $f'(x) = 3x^2 - 8x + 4$
b $f'(1) = -1$. This is the gradient of the tangent to the curve at the point $x = 1$.
c i 0   ii $y = 1$