Ch. 8 extra practice with some topics.

POLYNOMIALS 8.1.1 – 8.1.3

The chapter explores polynomial functions in greater depth. Students will learn how to sketch polynomial functions without using their graphing tool by using the factored form of the polynomial. In addition, they learn the reverse process: finding the polynomial equation from the graph. For further information see the Math Notes boxes in Lessons 8.1.1, 8.1.2, and 8.1.3.

Example 1

State whether or not each of the following expressions is a polynomial. If it is not, explain why not. If it is a polynomial, state the degree of the polynomial.

a. $-7x^4+\frac{2}{3}x^3+x^2-4.1x-6$

b. $8+3.2x^2-\pi x^5-61x^{10}$

c. $9x^3+4x^2-6x^{-1}+7^x$

d. $x(x^3+2)(x^4-4)$

A polynomial is an expression that can be written as the sum or difference of terms. The terms are in the form ax^n where a is any number called the coefficient of x, and n, the exponent, must be a whole number. The expression in part (a) is a polynomial. A coefficient that is a fraction $\left(\frac{2}{3}\right)$ is acceptable. The degree of the polynomial is the largest exponent on the variable, so in this case the degree is four. The expression in part (b) is also a polynomial, and its degree is ten. The expression in part (c) is not a polynomial for two reasons. First, the x^{-1} is not allowed because the exponents of the variable cannot be negative. The second reason is because of the 7^x. The variable cannot be a power in a polynomial. Although the expression in part (d) is not the sum or difference of terms, it can be written as the sum or difference of terms by multiplying the expression and simplifying. Doing this gives $x^8+2x^5-4x^4-8x$, which is a polynomial of degree 8.

Example 2

Without using your graphing tool, make a sketch of each of the following polynomials by using the orientation, roots, and degree.

a. $f(x)=(x+1)(x-3)(x-4)$

b. $y=(x-6)^2(x+1)$

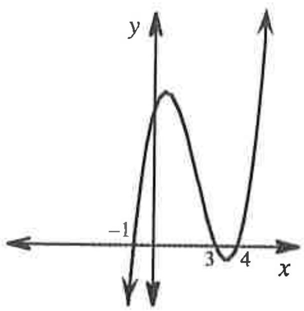
c. $p(x)=x(x+1)^2(x-4)^2$

d. $f(x)=-(x+1)^3(x-1)^2$

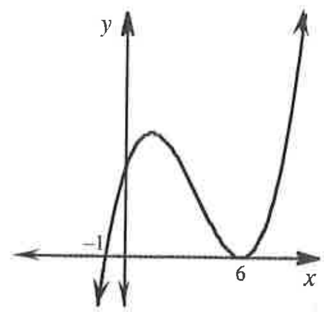
Through investigations, students learn a number of things about the graphs of polynomial functions. The roots of the polynomial are the x-intercepts, which are easily found when the polynomial is in factored form, as are all the polynomials above. Ask yourself the question: what values of x will make this expression equal to 0? The answer will give you the roots. In part (a), the roots of this third degree polynomial are $x = -1$, 3, and 4. In part (b), the roots of this third degree polynomial are 6 and -1. The degree of a polynomial tells you the maximum number of roots possible, and since this third degree polynomial has just two roots, you might ask where is the third root? $x = 6$ is called a double root, since that expression is squared and is thus equivalent to $(x-6)(x-6)$. The graph will just touch the x-axis at $x = 6$, and "bounce" off. The fifth degree polynomial in part (c) has three roots, 0, -1, and 4 with both -1 and 4 being double roots. The fifth degree polynomial in part (d) has two roots, -1 and 1, with 1 being a double root, and -1 being a triple root. The triple root "flattens" out the graph at the x-axis.

With the roots, we can sketch the graphs of each of these polynomials.

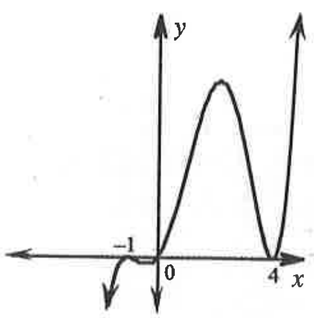
a.



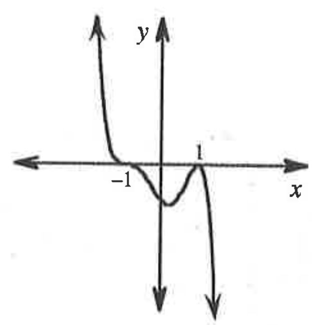
b.



c.



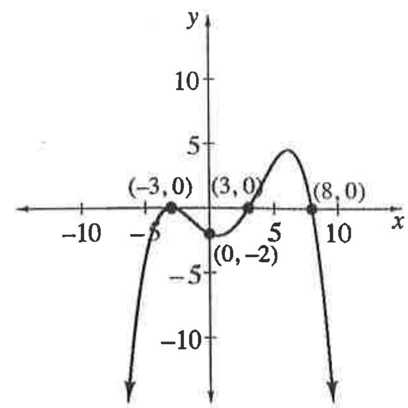
d.



Check that the roots fit the graphs. In addition, the graph in part (d) was the only one whose orientation was "flipped." Normally, a polynomial with an odd degree, starts off negative (as we move left of the graph) and ends up positive (as we move to the right). Because the polynomial in part (d) has a negative leading coefficient, its graph does the opposite.

Example 3

Write the exact equation of the graph shown at right.



From the graph we can write a general equation based on the orientation and the roots of the polynomial. Since the x-intercepts are -3, 3, and 8, we know $(x+3)$, $(x-3)$, and $(x-8)$ are factors. Also, since the graph touches at -3 and bounces off, $(x+3)$ is a double root, so we can write this function as $f(x)=a(x+3)^2(x-3)(x-8)$. We need to determine the value of a to have the exact equation.

Using the fact that the graph passes through the point $(0,-2)$, we can write:

$$\begin{aligned} -2 &= a(0+3)^2(0-3)(0-8) \\ -2 &= a(9)(-3)(-8) \\ -2 &= 216a \\ a &= -\tfrac{2}{216} = -\tfrac{1}{108} \end{aligned}$$

Therefore the exact equation is $f(x)=-\frac{1}{108}(x+3)^2(x-3)(x-8)$.

Problems

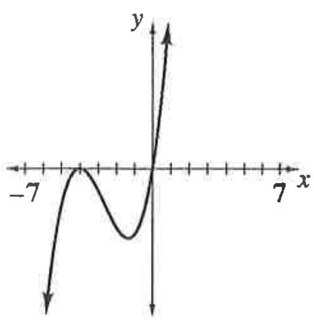
State whether or not each of the following is a polynomial function. If it is, give the degree. If it is not, explain why not.

1. $\frac{1}{8}x^7+4.23x^6-x^4-\pi x^2+\sqrt{2}x-0.1$
2. $45x^3-0.75x^2-\frac{3}{100}x+\frac{5}{x}+15$
3. $x(x+2)\left(6+\frac{1}{x}\right)$

Sketch the graph of each of the following polynomials.

4. $y=(x+5)(x-1)^2(x-7)$
5. $y=-(x+3)(x^2+2)(x+5)^2$
6. $f(x)=-x(x+8)(x+1)$
7. $y=x(x+4)(x^2-1)(x-4)$

Below are the complete graphs of some polynomial functions. Based on the shape and location of the graph, describe all the roots of the polynomial function, its degree, and orientation. Be sure to include information such as whether or not a root is a double or triple root.

8.


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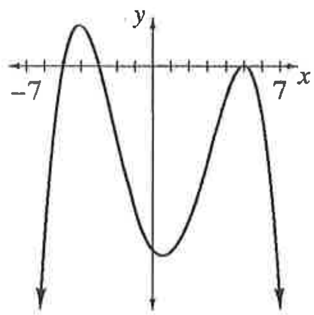

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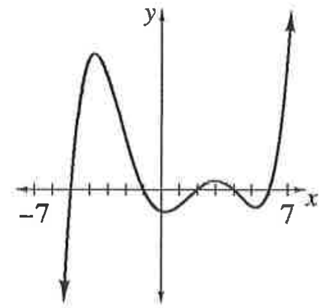
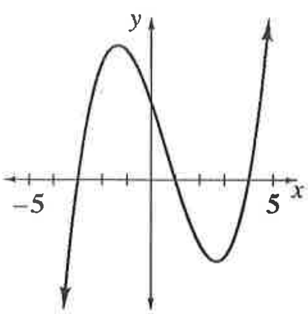
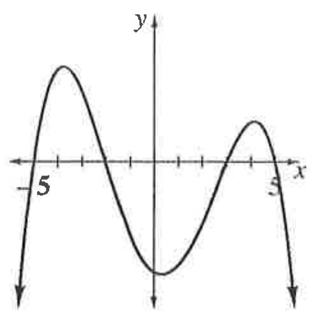
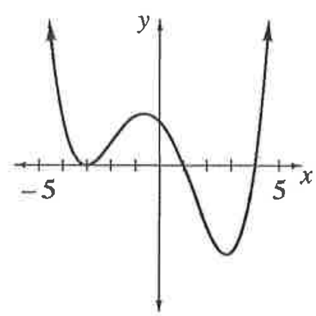


Using the graphs below and the given information, write the specific equation for each polynomial function.

11. y-intercept: $(0, 12)$


12. y-intercept: $(0, -15)$


13. y-intercept: $(0, 3)$


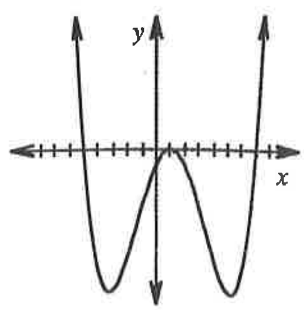
Answers

1. Yes, degree 7.

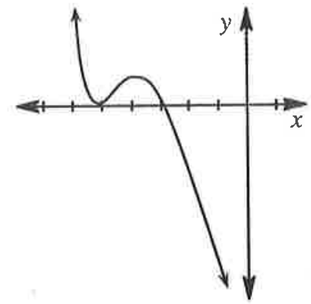
2. No. You cannot have x in the denominator.

3. No. When you multiply this out, you will still have x in the denominator.

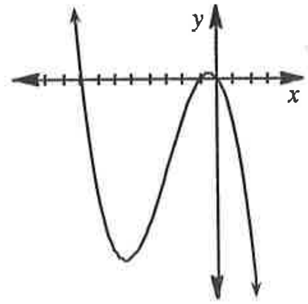
4. The roots are $x = -5, 1$, and 7 with $x = 1$ being a double root. Remember a double root is where the graph is tangent. This graph has a positive orientation.



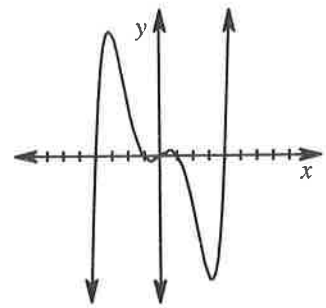
5. The roots are $x = -3$ and $x = -5$, which is a double root. The $x^2 + 2$ term does not produce any real roots since this expression cannot equal zero. The orientation is negative. The graph crosses the y-axis at $y = -150$.



6. This graph has negative orientation and the roots are $x = -8, -1$, and 0. Be sure to include $x = 0$ as a root.



7. $x^2 - 1$ gives us two roots. Since it factors to $(x+1)(x-1)$, the five roots are: $x = -4, -1, 0, 1$, and 4. The graph has a positive orientation.



8. A third degree polynomial (cubic) with one root at $x = 0$, and one double root at $x = -4$. It has a positive orientation.

9. A fourth degree polynomial with real roots at $x = -5$ and -3, and a double root at $x = 5$. It has a negative orientation.

10. A fifth degree polynomial with five real roots: $x = -5, -1, 2, 4$, and 6. It has a positive orientation.

11. $y = (x+3)(x-1)(x-4)$

12. $y = -0.1(x+5)(x+2)(x-3)(x-5)$

13. $y = \frac{1}{12}(x+3)^2(x-1)(x-4)$

COMPLEX NUMBERS

8.2.1 and 8.2.3

Students are introduced to the complex number system. Complex numbers arise naturally when trying to solve some equations such as $x^2+1=0$, which, until now, students thought had no solution. They see how the solution to this equation relates to its graph, its roots, and how imaginary and complex numbers arise in other polynomial equations as well. For further information see the Math Notes boxes in Lessons 8.2.1, 8.2.2, and 8.2.3.

Example 1

Solve the equation below using the Quadratic Formula. Explain what the solution tells you about the graph of the function.

$$2x^2-20x+53=0$$

As a quick review, the Quadratic Formula says: If $ax^2+bx+c=0$ then $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Here, $a=2, b=-20$, and $c=53$. Therefore,

$$\begin{aligned} x &= \frac{-(-20)\pm\sqrt{(-20)^2-4(2)(53)}}{2(2)} \\ &= \frac{20\pm\sqrt{400-424}}{4} \\ &= \frac{20\pm\sqrt{-24}}{4} \end{aligned}$$

We now have an expression with a negative under the radical. Until now, students would claim this equation has no solution. In fact, it has no **real** solution, but it does have a complex solution.

We define $i=\sqrt{-1}$ as an imaginary number. When we combine an imaginary number with a real number, we call it a complex number. Complex numbers are written in the form $a+bi$. Using i, we can simplify the answer above.

$$\begin{aligned} x &= \frac{20\pm\sqrt{-24}}{4} \\ &= \frac{20\pm\sqrt{-1\cdot4\cdot6}}{4} \\ &= \frac{20\pm2i\sqrt{6}}{4} \\ &= \frac{2\left(10\pm i\sqrt{6}\right)}{4} \\ &= \frac{10\pm i\sqrt{6}}{2} \end{aligned}$$

Because this equation has no real solutions, if we were to graph $y = 2x^2 - 20x + 53$ we would see a parabola that does not cross the x-axis. If we completed the square and put this into graphing form, we would get $y = 2(x-5)^2 + 3$. The vertex of this parabola is at $(5, 3)$, and since it open upwards, it will never cross the x-axis. You should verify this with your graphing tool.

Therefore, the graph of the function $y = 2x^2 - 20x + 53$ has no x-intercepts, but it does have two complex roots, $x = \frac{10 \pm i\sqrt{6}}{2}$. Recall that we said the degree of a polynomial function tells us the maximum number of roots. In fact the degree tells us the exact number of roots; some (or all) might be complex.

Example 2

Simplify each of the following expressions.

a. $3 + \sqrt{-16}$

b. $(3 + 4i) + (-2 - 6i)$

c. $(4i)(-5i)$

d. $(8 - 3i)(8 + 3i)$

Remember that $i = \sqrt{-1}$. Therefore, the expression in (a) can be written as $3 + \sqrt{-16} = 3 + 4\sqrt{-1} = 3 + 4i$. This is the simplest form; we cannot combine real and imaginary parts of the complex number. But, as is the case in part (b), we can combine real parts with real parts, and imaginary parts with imaginary parts: $(3 + 4i) + (-2 - 6i) = 1 - 2i$. In part (c), we can use the commutative rule to rearrange this expression: $(4i)(-5i) = (4 \cdot -5)(i \cdot i) = -20i^2$. However, remember that $i = \sqrt{-1}$, so $i^2 = (\sqrt{-1})^2 = -1$. Therefore, $-20i^2 = -20(-1) = 20$. Finally in part (d), we will multiply using methods we have used previously for multiplying binomials. You can use the Distributive Property or generic rectangles to compute this product.

$$\begin{aligned}(8 - 3i)(8 + 3i) &= 8(8) + 8(3i) - 3i(8) - 3i(3i) \\ &= 64 + 24i - 24i + 9 \\ &= 73\end{aligned}$$

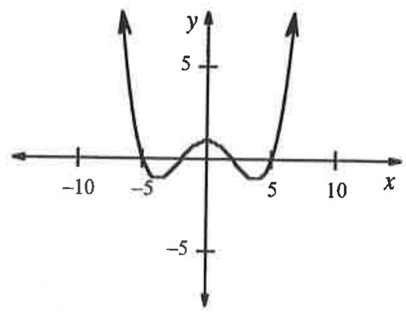
	8	$-3i$
8	64	$-24i$
$3i$	$24i$	9

The two expressions in part (d) are similar. In fact they are the same except for the middle sign. These two expressions are called **complex conjugates**, and they are useful when working with complex numbers. As you can see, multiplying a complex number by its conjugate produces a real number! This will always happen. Also, whenever a function with real coefficients has a complex root, it always has the conjugate as a root as well.

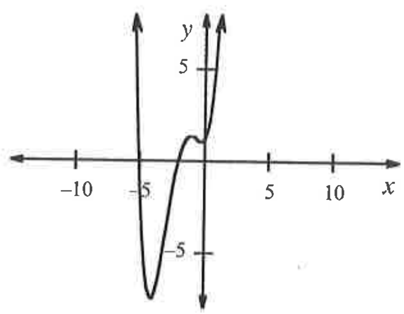
Example 3

Make a sketch of a graph of a polynomial function $p(x)$ so that $p(x) = 0$ would have only four real solutions. Change the graph so that it has two real and two complex solutions.

If $p(x) = 0$ is to have only four real solutions, then $p(x)$ will have four real roots. This will be a fourth degree polynomial that crosses the x-axis in exactly four different places. One such graph is shown at right.



In order for the graph to have only two real and two complex roots, we must change it so one of the "dips" does not reach the x-axis. One example is shown at right.



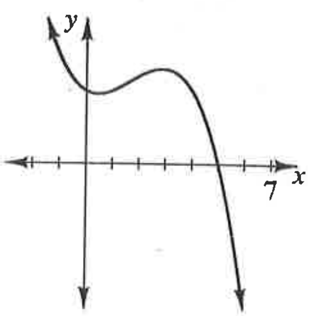
Problems

Simplify the following expressions.

1. $(6+4i)-(2-i)$
2. $8i-\sqrt{-16}$
3. $(-3)(4i)(7i)$
4. $(5-7i)(-2+3i)$
5. $(3+2i)(3-2i)$
6. $(\sqrt{3}-5i)(\sqrt{3}+5i)$

Below are the *complete* graphs of some polynomial functions. Based on the shape and location of the graph, describe all the roots of the polynomial function. Be sure to include information such as whether roots are double or triple, real or complex, etc.

7.



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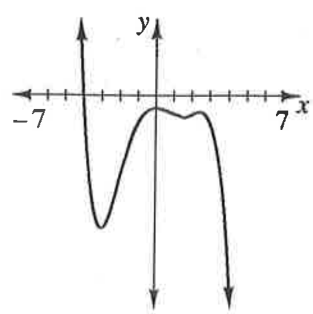

9. Write the specific equation for the polynomial function passing through the point (0, 5), and with roots $x = 5, x = -2$ and $x = 3i$.

Answers

1. $4+5i$

2. $4i$

3. 84

4. $11+29i$

5. 13

6. 28

7. A third degree polynomial with negative orientation and with one real root at $x=5$ and two complex roots.

8. A fifth degree polynomial with negative orientation and with one real root at $x=-4$ and four complex roots.

9. $y=-\frac{1}{18}\left(x^2-3x-10\right)\left(x^2+9\right)$

FACTORING POLYNOMIAL FUNCTIONS 8.3.1 and 8.3.3

Students learn to divide polynomials as one method for factoring polynomials of degrees higher than two. Through division and with two theorems, students are able to rewrite polynomials in a form more suitable for graphing. They can also easily find a polynomials' roots, both real and complex. For further information see the Math Notes boxes in Lessons 8.3.1, 8.3.2, and 8.3.3.

Example 1

Divide $x^3+4x^2-7x-10$ by $x+1$.

Students have learned to multiply polynomials using several methods, one of which is with generic rectangles. The generic rectangle is a method that works for polynomial division as well.

To find the product of two polynomials, students draw a rectangle and label the dimensions with the two polynomials. The area of the rectangle is the product of the two polynomials. For division, we start with the area and one dimension of the rectangle, and use the model to find the other dimension.

To review, consider the product $(x+2)(x^2+3x-7)$. We use the two expressions as the dimensions of a rectangle and calculate the area of each smaller part of the rectangle. In this case, the upper left rectangle has an area of x^3. The next rectangle to the right has an area of $3x^2$. We continue to calculate each smaller rectangle's area, and sum the collection to find the total area. The total area represents the product.

	x^2	$3x$	-7
x	x^3	$3x^2$	$-7x$
2	$2x^2$	$6x$	-14

Here, the total area is $x^3+3x^2-7x+2x^2+6x-14$, or x^3+5x^2-x-14 once it is simplified.

Now we will do the reverse of this process for our example. We will set up a rectangle that has a width of $x+1$ and an area of $x^3+4x^2-7x-10$. We have to move slowly, however, as we do not know what the length will be. We will add information to the figure gradually, adjusting as we go. The top left rectangle has an area equal to the highest-powered term: x^3. Now work backwards: If the area of this rectangle is x^3 and the side has a length of x, what does the length of the other side have to be? It would be an x^2. If we fill this piece of information above the upper left small rectangle we can use it to compute the area of the lower left rectangle.

x	x^3	
1		

	x^2	
x	x^3	
1		

	x^2	
x	x^3	
1	x^2	

The total area is $x^3 + 4x^2 - 7x - 10$, but we only have $1x^3$ and $1x^2$ so far. We will need to add $3x^2$ more to the total area (plus some other terms, but remember we are taking this one step at a time).

Once we have filled in the remaining "x^2" area, we can figure out the length of the top side. Remember that part of the left side has a length of x. This means that part of the top must have a length of $3x$.

Use this new piece of information to compute the area of the rectangle that is to the right of the x^3 rectangle, and then the small rectangle below that result.

	x^2	$3x$	
x	x^3	$3x^2$	
1	x^2	$3x$	

Our total area has a total of $-7x$, but we have only $3x$ so far. This means we will need to add $-10x$ more. Place this amount of area in the rectangle to the right of $3x^2$.

With this new piece of area added, we can compute the top piece's length and use it to calculate the area of the rectangle below the $-10x$. Note that our constant term in the total area is -10, which is what our rectangle has as well.

	x^2	$3x$	-10
x	x^3	$3x^2$	$-10x$
1	x^2	$3x$	-10

Therefore we can write $\frac{x^3+4x^2-7x-10}{x+1} = x^2 + 3x - 10$, or $x^3 + 4x^2 - 7x - 10 = (x+1)(x^2 + 3x - 10)$. Now that one of the terms is a quadratic, students can to factor it. Therefore, $x^3 + 4x^2 - 7x - 10 = (x+1)(x+5)(x-2)$.

Example 2

Factor the polynomial and find all its roots.

$$P(x) = x^4 + x^2 - 14x - 48$$

Students learn the Integral Zero Theorem, which says that zeros, or roots of this polynomial, must be factors of the constant term. This means the possible real roots of this polynomial are $\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm16, \pm24$, or ±48. In this case there are 20 possible roots to check! We can check them in a number of ways. One method is to divide the polynomial by the corresponding binomial expression (for instance, if -1 is a root, we divide the polynomial by $(x + 1)$ to see if it is a factor. Another method is to substitute each zero into the polynomial to see which of them, if any, make the polynomial equal to zero. We will still have to divide by the corresponding expression once we have the root, but it will mean less division in the long run.

Substituting values for $P(x)$, we get:

$$\begin{aligned} P(1) &= (1)^4 + (1)^2 - 14(1) - 48 \\ &= 1 + 1 - 14 - 48 \\ &= -60 \end{aligned}$$

$$\begin{aligned} P(-1) &= (-1)^4 + (-1)^2 - 14(-1) - 48 \\ &= 1 + 1 + 14 - 48 \\ &= -32 \end{aligned}$$

$$\begin{aligned} P(2) &= (2)^4 + (2)^2 - 14(2) - 48 \\ &= 16 + 4 - 28 - 48 \\ &= -56 \end{aligned}$$

$$\begin{aligned} P(-2) &= (-2)^4 + (-2)^2 - 14(-2) - 48 \\ &= 16 + 4 + 28 - 48 \\ &= 0 \end{aligned}$$

We can keep going, but we just found a root, $x = -2$. Therefore, $x + 2$ is a factor of the polynomial. Now we can divide the polynomial by this factor to find the other factors.

	x^3	$-2x^2$	$5x$	-24
x	x^4	$-2x^3$	$5x^2$	$-24x$
2	$2x^3$	$-4x^2$	$10x$	-48

This other factor, however, is degree three, still too high to use easier methods of factoring. Therefore we must use the Integral Zero Theorem again, and find another zero from the list ±1, $\pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24$. We can start where we left off, but now using $Q(x) = x^3 - 2x^2 + 5x - 24$, a simpler polynomial to evaluate.

$$\begin{aligned} Q(3) &= (3)^3 - 2(3)^2 + 5(3) - 24 \\ &= 27 - 18 + 15 - 24 \\ &= 0 \end{aligned}$$

It is certainly helpful to find a root so quickly! Since $x = 3$ is a root, $x - 3$ is a factor. Therefore, we can divide again.

	x^2	x	8
x	x^3	x^2	$8x$
-3	$-3x^2$	$-3x$	-24

$$x^2 + x + 8 = 0$$

$$\begin{aligned} x &= \frac{-1 \pm \sqrt{1^2 - 4(1)(8)}}{2(1)} \\ &= \frac{-1 \pm \sqrt{1-32}}{2} \\ &= \frac{-1 \pm \sqrt{-31}}{2} \\ &= \frac{-1 \pm i\sqrt{31}}{2} \end{aligned}$$

Now we have $P(x) = x^4 + x^2 - 14x - 48 = (x+2)(x-3)(x^2 + x + 8)$. Finally! The last polynomial is a quadratic (degree 2) so we can factor or use the Quadratic Formula. If you try factoring, you will not be successful, as this quadratic does not factor with integers. Therefore, we must use the Quadratic Formula to find the roots as shown at right.

Therefore, the original polynomial factors as:

$$P(x) = x^4 + x^2 - 14x - 48 = (x+2)(x-3)\left(x - \frac{-1+i\sqrt{31}}{2}\right)\left(x - \frac{-1-i\sqrt{31}}{2}\right)$$

Problems

1. Divide $3x^3 - 5x^2 - 34x + 24$ by $3x - 2$.
2. Divide $x^3 + x^2 - 5x + 3$ by $x - 1$.
3. Divide $6x^3 - 5x^2 + 5x - 2$ by $2x - 1$.

Factor the polynomials, keeping the factors real.

4. $f(x) = 2x^3 + x^2 - 19x + 36$
5. $g(x) = x^4 - x^3 - 11x^2 - 5x + 4$

Find all roots for each of the following polynomials.

6. $P(x) = x^4 - 2x^3 + x^2 - 8x - 12$
7. $Q(x) = x^3 - 14x^2 + 65x - 102$

Answers

1. $x^2 - x - 12$
2. $x^2 + 2x - 3$
3. $3x^2 - x + 2$
4. $f(x) = (x+4)(2x^2 - 7x + 9)$
5. $g(x) = (x+1)(x-4)(x^2 + 2x - 1)$
6. $x = -1, 3, 2i, -2i$
7. $x = 6, 4 + i, 4 - i$