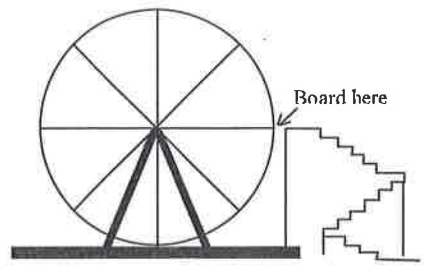
Ch. 7 extra practice with some topics

TRIGONOMETRIC FUNCTIONS 7.1.1 – 7.1.7

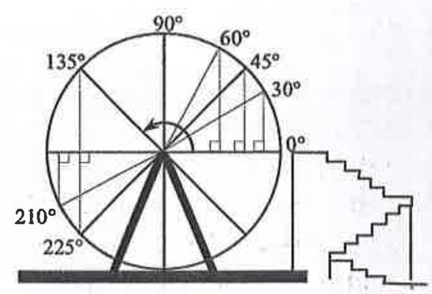
This chapter extends the students' knowledge of trigonometry. Students have already studied right triangle trigonometry, using sine, cosine and tangent with their calculators to find the lengths of unknown sides of triangles. Now students explore these same three trigonometry terms as functions. They are introduced to the unit circle, and they explore how the trigonometric functions are found within the unit circle. In addition, they learn a new way to measure angles using radian measure. For further information see the Math Notes boxes in Lessons 7.1.2, 7.1.5, 7.1.6, and 7.1.7.

Example 1

for background a good problem to understand



As Daring Davis stands in line waiting to ride the huge Ferris wheel, he notices that this Ferris wheel is not like any of the others he has ridden. First, this Ferris wheel does not board the passengers at the lowest point of the ride; rather, they board after climbing several flights of stairs, at the level of that wheel's horizontal axis. Also, if Davis thinks of the boarding point as a height of zero above that axis, then the maximum height above the boarding point that a person rides is 25 feet, and the minimum height below the boarding point is –25 feet. Use this information to create a graph that shows how a passenger's height on the Ferris wheel depends on the number of degrees of rotation from the boarding point of the Ferris wheel.



As the Ferris wheel rotates counterclockwise, a passenger's height above the horizontal axis increases, and reaches its maximum of 25 feet above the axis after 90° of rotation. Then the passenger's height decreases as measured from the horizontal axis, reaching zero feet after 180° of rotation, and continues to decrease as measured from the horizontal axis. The minimum height, –25 feet, occurs when the passenger has rotated 270°. After rotating 360°, the passenger is back where he started, and the ride continues.

To create this graph, we calculate the height of the passenger at various points along the rotation. These heights are shown using the grey line segments drawn from the passenger's location on the wheel perpendicular to the horizontal axis of the Ferris wheel. Note: Some of these values are easily filled in. At 0°, the height above the axis is zero feet. At 90°, the height is 25 feet.

Rotation, Degrees	0°	30°	45°	60°	90°	135°	180°	210°	225°	270°	315°	360°
Height, Feet	0				25		0			-25		0

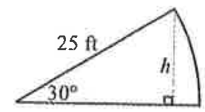
To complete the rest of the table we calculate the heights using right triangle trigonometry. We will demonstrate three of these values, 30°, 135°, and 225°, and allow you to verify the rest.

Each of these calculations involves focusing on the portion of the picture that makes a right triangle. For the 30° point, we look at the right triangle with a hypotenuse of 25 feet. (The radius of the circle is 25 feet because it is the maximum and minimum height the passenger reaches.) In this right triangle, we can use the sine function:

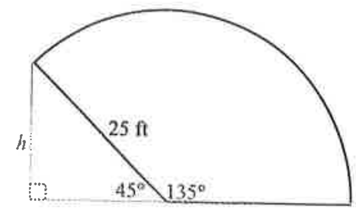
$$\sin 30° = \frac{h}{25}$$

$$25 \sin 30° = h$$

$$h = 12.5 \text{ feet}$$



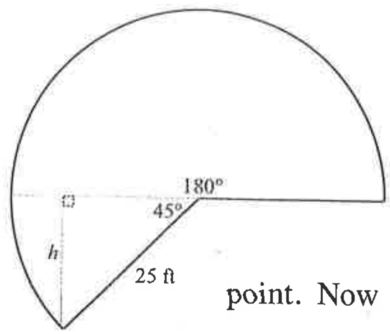
At the 135° mark, we use the right triangle on the "outside" of the curve. Since the angles are supplementary, the angle we use measures 45°.



$$\sin 45° = \frac{h}{25}$$

$$25 \sin 45° = h$$

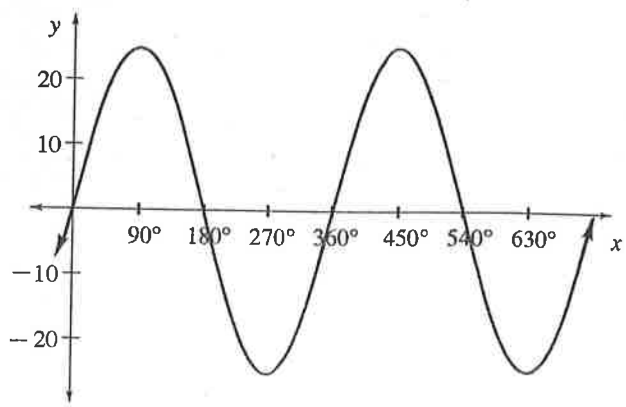
$$h \approx 17.68 \text{ feet}$$



At 225° (225 = 180 + 45), the triangle we use drops below the horizontal axis. We will use the 45° angle that is within the right triangle, so $h \approx -17.68$, using the previous calculation and changing the sign to represent that the rider is below the starting point. Now we can fill in all the values of the table.

Rotation, Degrees	0°	30°	45°	60°	90°	135°	180°	210°	225°	270°	315°	360°
Height, Feet	0	12.5	17.68	21.65	25	17.68	0	–12.5	–17.68	–25	–17.68	0

Plot these points and connect them with a smooth curve; your graph should look like the one at right. Note: This curve shows two revolutions of the Ferris wheel. This curve continues, repeating the cycle for each revolution of the Ferris wheel. It also represents a particular sine function: $y = 25 \sin x$.

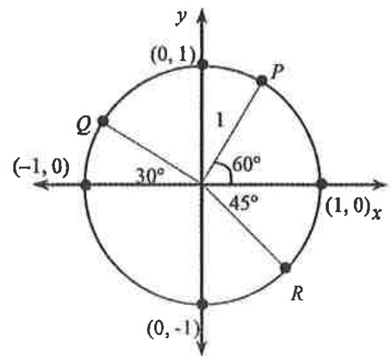


Example 2

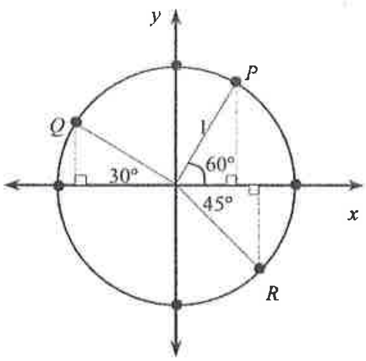
use methods in class → skip to page 80

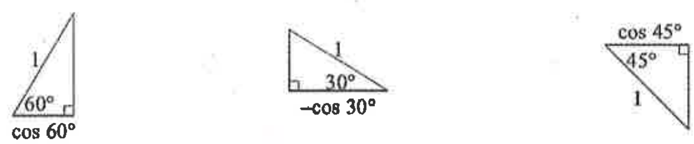
On a unit circle, represent and then calculate cos 60°, cos 150°, and cos 315°. Then graph $y = \cos x$.

The trigonometric functions ("trig" functions) arise naturally in circles as we saw with the first example. The simplest circle is a unit circle, that is, a circle of radius 1 unit, and it is this circle we often use with the trig functions.

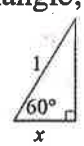


On the unit circle at right, several points are labeled. Point P corresponds to a 60° rotation, point Q corresponds to 150°, and R corresponds to 315°. We measure rotations from the point $(1, 0)$ counter-clockwise to determine the angle. If we create right triangles at each of these points, we can use the right triangle trig we learned in geometry to determine the lengths of the legs of the triangle. In the previous example, the height of the triangle was found using the sine. Here, the cosine will give us the length of the other leg of the triangle.



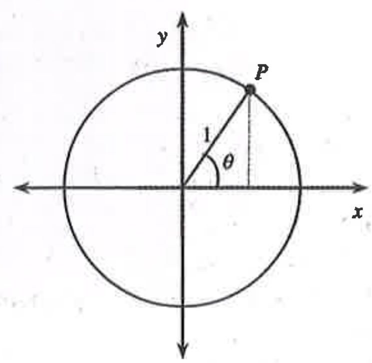


To fully understand why the length of the horizontal leg is labeled with "cosine," consider the triangle below. In the first triangle, if we labeled the short leg x, we would write:



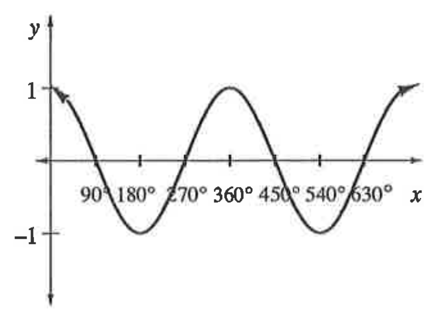
$$\cos 60° = \frac{x}{1}$$

$$x = \cos 60°$$



Therefore the length of the horizontal leg of the first triangle is cos 60°. Note: The second triangle representing 150°, lies in the second quadrant where the x-values are negative. Therefore cos 150° = –cos 30°. Check this on your calculator.

It is important to note what this means. On a unit circle, we can find a point P by rotating q degrees. If we create a right triangle by dropping a height from point P to the x-axis, the length of this height is always sin q. The length of the horizontal leg is always cos q. Additionally, this means that the coordinates of point P are (cos q, sin q). This is the power of using a unit circle: the coordinates of any point on the circle are found by taking the sine and cosine of the angle. The graph at right shows the cosine curve for two rotations around the unit circle.



Example 3

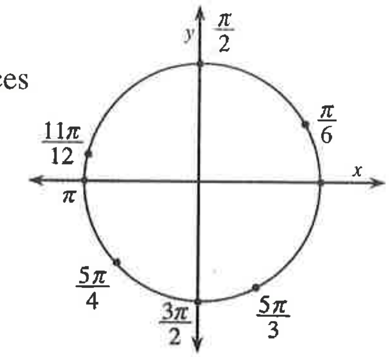
On a unit circle, find the points corresponding to the following radians. Then convert each angle given in radians to degrees.

great practice

a. $\frac{\pi}{6}$ b. $\frac{11\pi}{12}$ c. $\frac{5\pi}{4}$ d. $\frac{5\pi}{3}$

One radian is about 57°, but that is not the way to remember how to convert from degrees to radians. Instead, think of the unit circle, and remember that one rotation would be the same as traveling around the unit circle one circumference. The circumference of the unit circle is $C = 2\pi r = 2\pi(1) = 2\pi$. Therefore, one rotation around the circle, 360°, is the same as traveling 2π radians around the circle. Radians do not just apply to unit circles. A circle with any size radius still has 2π radians in a 360° rotation.

We can place these points around the unit circle in appropriate places without converting them. First, remember that 2π radians is the same point as a 360° rotation. That makes half of that, 180°, corresponds to π radians. Half of that, 90°, is $\frac{\pi}{2}$ radians. With similar reasoning, 270° corresponds to $\frac{3\pi}{2}$ radians. Using what we know about fractions allows us to place the other radian measures around the circle. For example, $\frac{\pi}{6}$ is one-sixth the distance to π.



Sometimes we want to be able to convert from radians to degrees and back. To do so, we can use a ratio of $\frac{\text{radians}}{\text{degrees}}$. To convert $\frac{\pi}{6}$ radians to degrees we create a ratio, and solve for x. We will use $\frac{\pi}{180}$ as a simpler form of $\frac{2\pi}{360}$. Therefore $\frac{\pi}{6}$ radians is equivalent to 30°. Similarly, we can convert the other angles above to degrees:

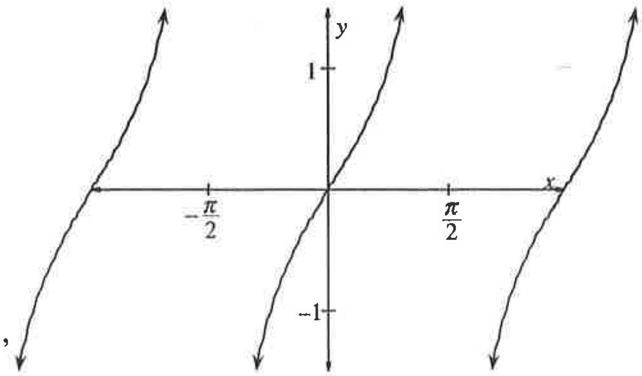
$$\frac{\pi}{180^\circ} = \frac{\pi/6}{x}$$
$$x\pi = 180\left(\frac{\pi}{6}\right)$$
$$x\pi = 30\pi$$
$$x = 30^\circ$$

$$\frac{\pi}{180^\circ} = \frac{11\pi/12}{x^\circ} \qquad \frac{\pi}{180^\circ} = \frac{5\pi/4}{x^\circ} \qquad \frac{\pi}{180^\circ} = \frac{5\pi/3}{x^\circ}$$
$$x^\circ\pi = 180^\circ(\tfrac{11\pi}{12}) \qquad x^\circ\pi = 180^\circ(\tfrac{5\pi}{4}) \qquad x^\circ\pi = 180^\circ(\tfrac{5\pi}{3})$$
$$x^\circ\pi = 165^\circ\pi \qquad x^\circ\pi = 225^\circ\pi \qquad x^\circ\pi = 300^\circ\pi$$
$$x = 165^\circ \qquad x = 225^\circ \qquad x = 300^\circ$$

can skip

Example 4

Graph $T(\theta) = \tan\theta$. Explain what happens at the points $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$. Why does this happen?



As with the graphs of $S(\theta) = \sin\theta$ and $C(\theta) = \cos\theta$, $T(\theta) = \tan\theta$ repeats, that is, it is cyclic. The graph does not, however, have the familiar hills and valleys the other two trig functions display. This graph, shown at right, resembles in part the graph of a cubic such as $f(x) = x^3$. However, it is *not* a cubic, which is clear from the fact that it has asymptotes and repeats. At $\theta = \frac{\pi}{2}$, the graph approaches a vertical asymptote. This also occurs at $\theta = -\frac{\pi}{2}$, and because the graph is cyclic, it happens repeatedly at $\theta = \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$. In fact, it happens at all values of θ of the form $\frac{(2k-1)\pi}{2}$ for all integer values of k (odd values).

The real question is, *why* does this asymptote occur at these points? Recall that $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Every point where $\cos\theta = 0$, this function is undefined (we cannot have zero in the denominator). So at each point where $\cos\theta = 0$, the function $T(\theta) = \tan\theta$ is also undefined. Examining the graph of $C(\theta) = \cos\theta$, we can see that this graph is zero (crosses the x-axis) at the same type of points as above: $\frac{(2k-1)\pi}{2}$ for all integer values of k.

Problems

Graph each of the following trig equations.

1. $y = \sin x$ 2. $y = \cos x$ 3. $y = \tan x$

Find each of the following values without using a calculator, but by using what you know about right triangle trigonometry, the unit circle, and special right triangles.

all

4. $\cos(180°)$ 5. $\sin(360°)$ 6. $\tan(45°)$

7. $\cos(-90°)$ 8. $\sin(150°)$ 9. $\tan(240°)$

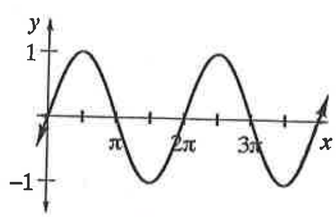
Convert each of the angle measures.

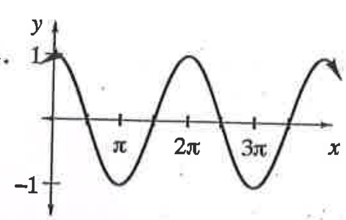
all

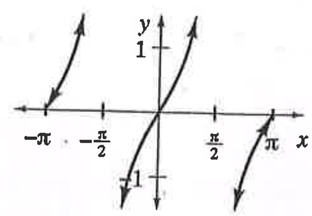
10. $60°$ to radians 11. $170°$ to radians 12. $315°$ to radians

13. $\frac{\pi}{15}$ radians to degrees 14. $\frac{13\pi}{8}$ radians to degrees 15. $-\frac{3\pi}{4}$ radians to degrees

Answers

1. 

2. 

3. 

4. -1 5. 0 6. 1

7. 0 8. $\frac{1}{2}$ 9. $\sqrt{3}$

10. $\frac{\pi}{3}$ radians 11. $\frac{17\pi}{18}$ radians 12. $\frac{7\pi}{4}$ radians

13. $12°$ 14. $292.5°$ 15. $-135°$

TRANSFORMING TRIG FUNCTIONS 7.2.1 – 7.2.4

Students apply their knowledge of transforming parent graphs to the trigonometric functions. They will generate general equations for the family of sine, cosine and tangent functions, and learn about a new property specific to cyclic functions called the period. The Math Notes box in Lesson 7.2.4 illustrates the different transformations of these functions.

Example 1

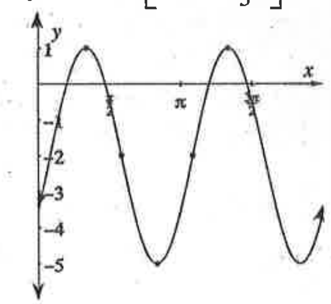
For each of the following equations, state the amplitude, number of cycles in 2π, horizontal shift, and vertical shift of the graph. Then graph each on equation separate axes.

$$y = 3\cos\left[2(x - \tfrac{\pi}{3})\right] - 2 \qquad\qquad y = -\sin\left[\tfrac{1}{4}(x + \pi)\right] + 1$$

The general form of a sine function is $y = a\sin\left[b(x-h)\right]+k$. Some of the transformations of trig functions are standard ones that students learned in Chapter 2. The a will determine the orientation, in this case, whether it is in the standard form, or if it has been reflected across the x-axis. With trigonometric functions, a also represents the amplitude of the function: half of the distance the function stretches from the maximum and minimum points vertically. As before, h is the horizontal shift, and k is the vertical shift. This leaves just b, which tells us about the period of the function. The graphs of $y = \sin\theta$ and $y = \cos\theta$ each have a period of 2π, which means that one cycle (before it repeats) has a length of 2π. However, b affects this length since b tells us the number of cycles that occur in the length 2π.

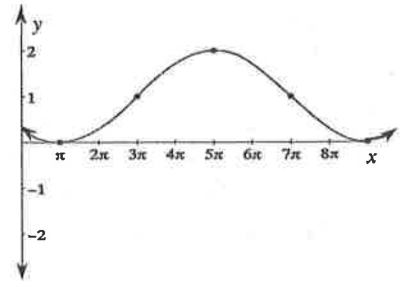
The first function, then, has an amplitude of 3, and since this is positive, it is not reflected across the x-axis. The graph is shifted horizontally to the right $\frac{\pi}{3}$ units, and shifted down (vertically) 2 units. The 2 before the parentheses tells us it does two cycles in 2π units. If the graph does two cycles in 2π units, then the length of the period is π units. The graph of this function is shown at right.

$y = 3\cos\left[2(x - \tfrac{\pi}{3})\right] - 2$



The second function has an amplitude of 1, but it is reflected across the x-axis. It is shifted to the left π units, and shifted up 1 unit. Here we see that within a 2π span, only one-fourth of a cycle appears. This means the period is four times as long as normal, or is 8π. The graph is shown at right.

$y = -\sin\left[\tfrac{1}{4}(x + \pi)\right] + 1$



Example 2

For the Fourth of July parade, Vicki decorated her tricycle with streamers and balloons. She stuck one balloon on the outside rim of one of her back tires. The balloon starts at ground level. As she rides, the height of balloon rises up and down, sinusoidally (that is, a sine curve). The diameter of her tire is 10 inches.

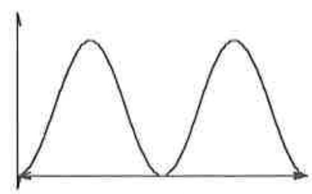
a. Sketch a graph showing the height of the balloon above the ground as Vicki rolls along.

b. What is the period of this graph?

c. Write the equation of this function.

d. Use your equation to predict the height of the balloon after Vicki has traveled 42 inches.

This problem is similar to the Ferris Wheel example at the beginning of this chapter. The balloon is rising up and down just as a sine or cosine curve rises up and down. A simple sketch is shown at right.

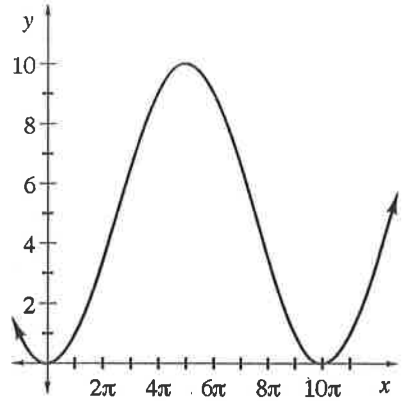


The balloon begins next to the ground and as the tricycle wheel rolls, the balloon rises to the top of the wheel, then comes back down. If we let the ground represent the x-axis, the balloon is at its highest point when it is at the top of the wheel, a distance of one wheel's height or diameter, 10 inches. So now we know that the distance from the highest point to the lowest point is 10. The amplitude is half of this distance, 5.

To determine the period, we need to think about the problem. The balloon starts at ground level, rises as the wheel rolls and comes down again to the ground. What has happened when the balloon returns to the ground? The wheel has made one complete revolution. How far has the wheel traveled then? It has traveled the distance of one circumference. The circumference of a circle with diameter 10 inches is 10π inches. Therefore the period of this graph is 10π.

To get the equation for this graph we need to make some decisions. The graphs of sine and cosine are similar. In fact, one is just the other shifted 90° (or $\frac{\pi}{2}$ radians). At this point, we need to decide if we want to use sine or cosine to model this data. Either one will work but the answers will look different. Since the graph starts at the lowest point and not in the middle, this suggests that we use cosine. (Yes, cosine starts at the highest point but we can multiply by a negative to flip the graph over and start at the lowest point.) We also know the amplitude is 5 and there is no horizontal shift. All of this information can be written in the equation as $y = -5\cos[bx] + k$. We can determine k by remembering that we set the x-axis as the ground. This implies the graph is shifted up 5 units. To determine the number of cycles in 2π (that is, b), recall that we found that the period of this graph is 10π. Therefore $\frac{2\pi}{10} = \frac{1}{5}$ of the curve appears within the 2π span. Finally, pulling everything together we can write $y = -5\cos\left[\frac{1}{5}x\right] + 5$, and is shown in the following graph.

$y = -5\cos\left[\frac{1}{5}x\right] + 5$



Note: If you decided to use the sine function for this data, you must realize that the graph is shifted to the right $\frac{10\pi}{4}$ units. One equation that gives this graph is $y = 5\sin\left[\frac{1}{5}(x - \frac{10\pi}{4})\right] + 5$. There are other equations that work, so if you do not get the same equation as shown here, graph yours and compare.

To find the height of the balloon after Vicki rides 42 inches, we substitute 42 for x in the equation.

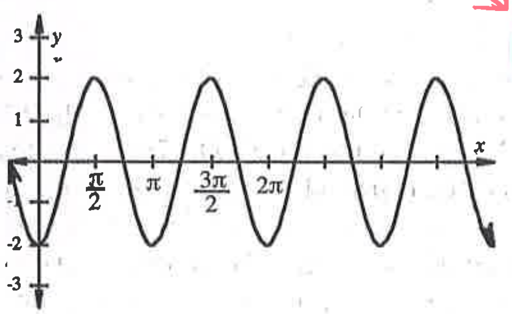
If you do not get this answer, make sure your calculator is in radians!

$$y = -5\cos\left[\frac{1}{5}\cdot 42\right] + 5$$
$$\approx -5\cos(8.4) + 5$$
$$\approx 7.596 \text{ inches}$$

Problems

Examine each graph below. For each one, draw a sketch of one cycle, then give the amplitude and the period.

1.



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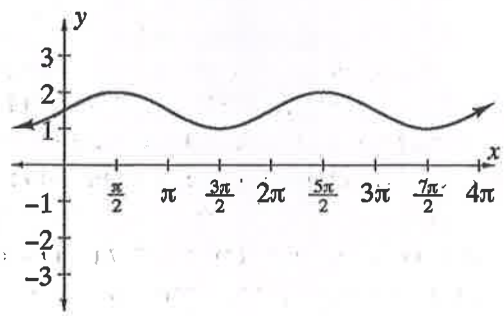
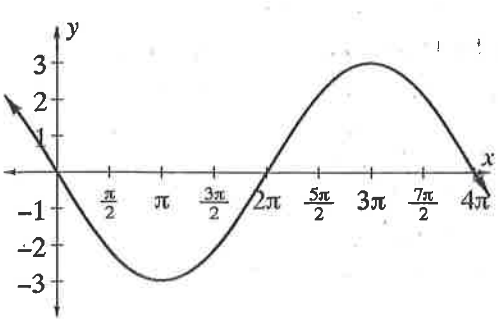
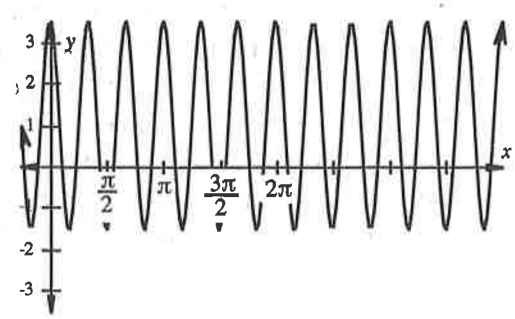


3.



4.



For each equation listed below, state the amplitude and period.

5. $y = 2\cos(3x) + 7$

6. $y = \frac{1}{2}\sin x - 6$

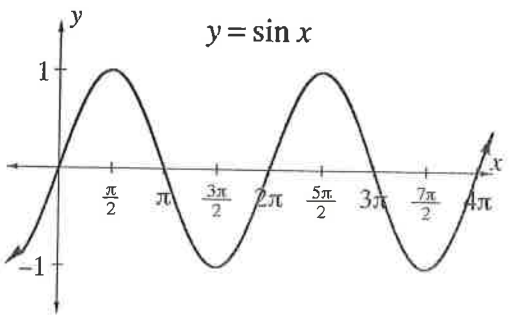
7. $f(x) = -3\sin(4x)$

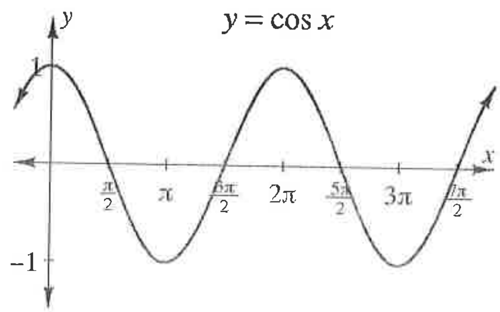
8. $y = \sin\left[\frac{1}{3}x\right] + 3.5$

9. $f(x) = -\cos x + 2\pi$

10. $f(x) = 5\cos(x-1) - \frac{1}{4}$

Below are the graphs of $y = \sin x$ and $y = \cos x$.





Use them to sketch the graphs of each of the following equations and functions by hand. Use your graphing calculator to check your answer.

11. $y = -2\sin(x + \pi)$

12. $f(x) = \frac{1}{2}\sin(3x)$

13. $f(x) = \cos\left(4\left(x - \frac{\pi}{4}\right)\right)$

14. $y = 3\cos\left(x + \frac{\pi}{4}\right) + 3$

15. $f(x) = 7\sin\left(\frac{1}{4}x\right) - 3$

16. A wooden water wheel is next to an old stone mill. The water wheel makes ten revolutions every minute, dips down two feet below the surface of the water, and at its highest point is 18 feet above the water. A snail attaches to the edge of the wheel when the wheel is at its lowest point and rides the wheel as it goes round and around. As time passes, the snail rises up and down, and in fact, the height of the snail above the surface of the water varies sinusoidally with time. Use this information to write the particular equation that gives the height of the snail over time.

17. To keep baby Cristina entertained, her mother often puts her in a Johnny Jump Up. It is a seat on the end of a strong spring that attaches in a doorway. When Mom puts Cristina in, she notices that the seat drops to just 8 inches above the floor. One and a half seconds later (1.5 seconds), the seat reaches its highest point of 20 inches above the ground. The seat continues to bounce up and down as time passes. Use this information to write the particular equation that gives the height of baby Cristina's Johnny Jump Up seat over time. (Note: You can start the graph at the point where the seat is at its lowest.)

Answers

1.  Amplitude is 2, period is π.

2.  Amplitude 0.5, period 2π.

3. The graph shows one cycle already. Amplitude is 3 and period is 4π.

4.  Amplitude is 2.5, period is $\frac{\pi}{3}$.

5. Amplitude: 2, period: $\frac{2\pi}{3}$.

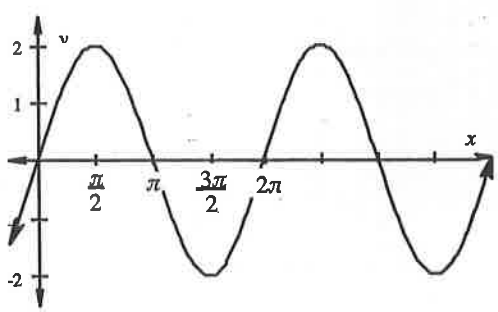
6. Amplitude: , period: 2π.

7. Amplitude: 3, period: $\frac{\pi}{2}$.

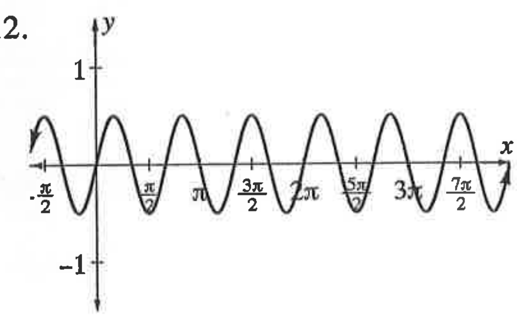
8. Amplitude: 1, period: 6π.

9. Amplitude: 1, period: 2π.

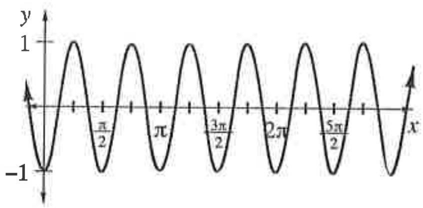
10. Amplitude: 5, period: 2π.

11. 

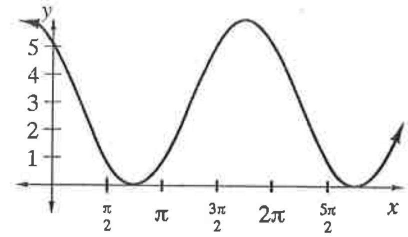
Surprised? The negative flips it over, but the "+ π" shifts it right back to how it looks originally!

12. 

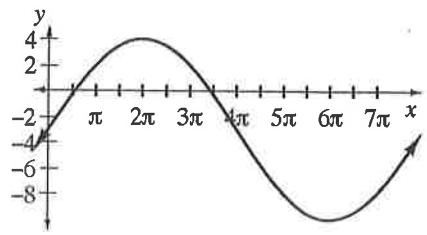
13.



14.



15.



16. $y=-10\cos(\frac{1}{10}x)+8$, and there are other possible equations which will work.

17. $y=-6\cos(\frac{2\pi}{3}x)$ works if we let the graph be symmetric about the x-axis. The x-axis does not have to represent the ground. If you let the x-axis represent the ground, you equation might look like $y=-6\cos(\frac{2\pi}{3}x)+14$.