SYSTEMS OF THREE EQUATIONS

This chapter begins with students using technology to explore graphing in three dimensions. By using strategies that they used for graphing in two dimensions, students extend their skills to plotting points as well as graphing planes represented by equations with three variables. This leads to multiple planes intersecting, and using algebra to find either the equation of the line of the intersecting planes, or the point of intersection for the system of three equations and three unknowns. For more information, see the Math Notes boxes in Lessons 6.1.2 and 6.1.4.

SKIP

Example 1

Graph the 3-D point and equation below.

$$(3, 4, 5) \qquad\qquad 2x + 3y + 4z = 24$$

Although we live in a three-dimensional world, visualizing three-dimensional objects on a two-dimensional piece of paper can be difficult. In class, students used the computer to help them visualize the graphs. (You an access this software at home at http://www.cpm.org/students/technology/CCA2/. Click on the desired item from Chapter 6.) Students begin by plotting points on axes as shown at right. As with plotting points in two dimensions, each number of the coordinate tells us how far to move along the x-axis, then the y-axis, and finally the z-axis. Here we are only showing the positive direction for each axis; these axes extend in the negative direction as well.
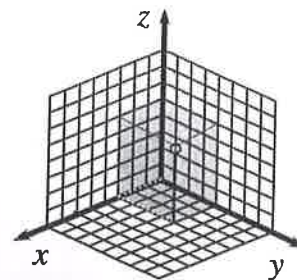
To plot the point $(3, 4, 5)$, we move along the x-axis three units, along the direction of the y-axis four units, and then five units along the z-axis. To help illustrate this, the point is marked with a circle. The path to the point is shown with a dotted line along the x and y directions, and a solid line to show the rise in the z direction. It might help students to think of the point as the corner on a box, farthest from the origin. This imaginary box is lightly shaded in above.

To graph the equation with three variables on the three dimensional graph, it is helpful to find where it crosses each axis. We do this by letting the different variables equal zero, which allows us to find the x-, y-, and z-intercepts.
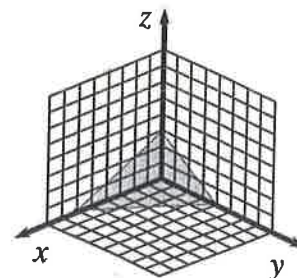
$$2x + 3y + 4z = 12$$
$$x = 0, \ y = 0 \ \Rightarrow \ 0 + 0 + 4z = 12, \ z = 3$$
$$x = 0, \ z = 0 \ \Rightarrow \ 0 + 3y + 0 = 12, \ y = 4$$
$$y = 0, \ z = 0 \ \Rightarrow \ 2x + 0 + 0 = 12, \ x = 6$$

By plotting these intercepts we see how the plane slices through this quadrant of space. The shaded plane continues; it does not stop at the edges of the triangle.

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Example 2

Solve the following system of three equations and three unknowns. Explain what your solution says about the graphs of each equation.

$$2x + y - 3z = 13$$
$$x - 3y + z = -21$$
$$-2x + y + 4z = -7$$

Before beginning, it is helpful to recall how students solved two equations with two unknowns. If an equation was written in $y =$ form, students substituted the expression for y into the other equation. That method will not work as easily with three equations. Other times with two equations and two unknowns, students would add or subtract two equations to make a variable disappear. Sometimes they needed to multiply an equation by some number before adding or subtracting. In either procedure, the goal was the same: to eliminate a variable.

$$
\begin{array}{r}
2x + y - 3z = 13 \\
-2x + y + 4z = -7 \\
\hline
2y + z = 6
\end{array}
$$

We will use the same approach here. By adding the first and third equation above, we eliminate the x. The problem is, we still have two variables. The goal now is to find another pair of equations from which to eliminate x. There are different ways to do this. Here, we will multiply the second equation by two, then add the result to the third equation.

$$
2(x - 3y + z) = 2(-21) \implies
\begin{array}{r}
2x - 6y + 2z = -42 \\
-2x + y + 4z = -7 \\
\hline
-5y + 6z = -49
\end{array}
$$

Now that we have two equations with two unknowns, we will use this simpler problem to solve for y and z.

$$
6 \cdot (2y + z = 6) \implies
\begin{array}{r}
12y + 6z = 36 \\
-(-5y + 6z = -49) \\
\hline
17y = 85 \\
y = 5
\end{array}
\qquad
\begin{array}{l}
y = 5 \implies 2y + z = 6 \\
 2(5) + z = 6 \\
 10 + z = 6 \\
 z = -4
\end{array}
$$

Now that we know what y and z are, we can substitute them back into any one of our original equations to determine the value of x.

$$
y = 5,\ z = -4 \implies
\begin{array}{l}
x - 3y + z = -21 \\
x - 3(5) + (-4) = -21 \\
x - 15 - 4 = -21 \\
x = -2
\end{array}
$$

Therefore the solution to this system is $(-2, 5, -4)$ which tells us all three planes intersect in one point.

Example 3

Pizza Planet sells three sizes of combination pizzas.

Small (8" diameter)	$8.50
Medium (10" diameter)	$11.50
Large (13" diameter)	$17.50

Assume that the price of the pizza can be modeled with a quadratic function, with the price dependent on the diameter of the pizza. Use the information to write three data points, and determine an equation representing the data points. If *Pizza Planet* is considering selling an Extra Large combination pizza, with an 18" diameter, what should such a pizza cost? If they wanted to sell a combination pizza for $50.00, how big would it have to be to fit with the rest of the price data for the pizzas?

If we let x represent the diameter of the pizza in inches, and y represent the cost of the pizza in dollars, the three data points are $(8, 8.50)$, $(10, 11.50)$, and $(13, 17.50)$. We use these three points in the general equation for a quadratic, $y = ax^2 + bx + c$. Our goal is to determine the appropriate values for a, b, and c so that the graph of the quadratic equation passes through the three data points. To be able to do this, we will need to solve three equations with three unknowns.

First, we substitute the data points into the general equation.

$(8, 8.50) \implies y = ax^2 + bx + c$
$$8.50 = a(8)^2 + b(8) + c$$
$$8.50 = 64a + 8b + c$$

$(10, 11.50) \implies y = ax^2 + bx + c$
$$11.50 = a(10)^2 + b(10) + c$$
$$11.50 = 100a + 10b + c$$

$(13, 17.50) \implies y = ax^2 + bx + c$
$$17.50 = a(13)^2 + b(13) + c$$
$$17.50 = 169a + 13b + c$$

This gives us the three equations shown at right. (For further reference, the equations are numbered.) This is now similar to the last example; we must solve three equations with three unknowns. Our unknowns here are a, b, and c, rather than x, y, and z.

$$8.50 = 64a + 8b + c \quad (1)$$
$$11.50 = 100a + 10b + c \quad (2)$$
$$17.50 = 169a + 13b + c \quad (3)$$

To begin we will eliminate c by subtracting pairs of equations. Equation (2) minus equation (1) gives $3 = 36a + 2b$; equation (3) minus equation (2) gives $6 = 69a + 3b$. Now we are back to the more familiar two equations with two unknowns. To solve them for a and b, we will multiply the first by −3 and the second by 2 then add the results.

$$3 = 36a + 2b \quad \overset{\times(-3)}{\implies} \quad -9 = -108a - 6b$$

$$6 = 69a + 3b \quad \overset{\times 2}{\implies} \quad \underline{12 = 138a + 6b}$$

$$3 = 30a$$

$$a = \tfrac{3}{30} = \tfrac{1}{10}$$

Now that we know the value of a, we substitute it back into the first (or second) equation to find the value of b.

$$a = \tfrac{1}{10} \Rightarrow 3 = 36a + 2b$$
$$3 = 36\left(\tfrac{1}{10}\right) + 2b$$
$$3 = 3.6 + 2b$$
$$-0.6 = 2b$$
$$b = -0.3 = -\tfrac{3}{10}$$

Lastly, we use the values for a and b to find c. We can use any of the three equations (1), (2), or (3). To keep it simple, we will use (1).

$$a = \tfrac{1}{10}, \; b = -\tfrac{3}{10} \quad \Rightarrow \quad 8.50 = 64a + 8b + c$$
$$8.50 = 64\left(\tfrac{1}{10}\right) + 8\left(-\tfrac{3}{10}\right) + c$$
$$8.50 = 6.4 - 2.4 + c$$
$$8.50 = 4 + c$$
$$c = 4.50$$

Note: Any of the three original equations would have worked, and in fact, equation (2) would have eliminated the fractions and decimals from our work.

Now that we have found a, b, and c, we can write the equation that models this data: $y = \tfrac{1}{10}x^2 - \tfrac{3}{10}x + 4.50$. We use this equation to determine the cost of a combination pizza with an 18-inch diameter.

$$x = 18 \quad \Rightarrow \quad y = \tfrac{1}{10}x^2 - \tfrac{3}{10}x + 4.50$$
$$y = \tfrac{1}{10}(18)^2 - \tfrac{3}{10}(18) + 4.50$$
$$y = 32.4 - 5.4 + 4.50$$
$$y = \$31.50$$

Therefore an 18-inch combination pizza should cost $31.50. How large should a $50.00 pizza be to fit with this data? To answer this we must let $y = 50$, and solve for x. The solution will require solving a quadratic equation. Although the students know several ways to solve quadratics, the best approach here is to us the Quadratic Formula. To begin, we multiply everything by 10 to eliminate the fractions and the decimals.

$$50 = \tfrac{1}{10}x^2 - \tfrac{3}{10}x + 4.50$$
$$500 = x^2 - 3x + 45$$
$$x^2 - 3x - 455 = 0$$
$$x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-455)}}{2(1)}$$
$$x = \frac{3 \pm \sqrt{9 + 1820}}{2}$$
$$x \approx \frac{3 \pm 42.77}{2}$$
$$x \approx 22.89$$

Therefore, to sell a pizza for $50.00, *Pizza Planet* should make the diameter of the pizza approximately 22.89 inches. A 23-inch diameter would surely suffice!

Problems

Solve each of the following systems of equations for x, y, and z. Explain what the answer tells you about the graphs of the equations.

1. $3x - 2y + z = 3$
   $5x + y + 2z = 8$
   $-3x - y + 3z = -22$

2. $-4x - 6y + 5z = 21$
   $3x + 4y - 2z = -15$
   $-7x - 5y + 3z = 15$

3. $3x + 4z = 19$
   $3y + 2z = 8$
   $4x - 5y = 7$

4. $9x + 6y - 12z = 14$
   $3x + 2y - 4z = -11$
   $x + y + z = 1$

5. $\frac{x}{3} + \frac{y}{4} + \frac{z}{2} = 24$
   $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 29$
   $\frac{x}{4} + \frac{y}{2} + \frac{z}{3} = 25$

6. $21x - 7y + 14z = 70$
   $15x - 5y + 10z = 50$
   $-3x + y - 2z = -10$

7. Find the equation of the parabola passing through the three points $(-2, -32)$, $(0, -10)$, and $(2, -12)$.

8. Find the equation of the parabola passing through the three points $(2, 81)$, $(7, 6)$, and $(10, 33)$.

9. While reading a recent study done on people of various ages, you notice a trend. The study counted the number of times in a 24 hour period that the people misunderstood or misinterpreted a statement, comment, or question. The study offers the numbers shown in the table below.

Age (years)	Misunderstandings or Misinterpretations
20	44
30	28
40	20

You believe that the number of misunderstandings should reach a minimum at some age then go up again for very old people. Therefore, you assume that a quadratic function will best model this data. Find the equation that best fits this data. Use your equation to predict how many times an 80-year old person will misunderstand or misinterpret a statement, comment, or question. What about a one-year old? Who understands the most?

10. In archery, the arrow appears to travel in a straight line when it is released. However, the arrow will actually travel upward slightly before curving back down toward the earth. For a particular archer, the arrow starts at 5.4 feet above the ground. After 0.3 seconds, the arrow is 5.5 feet above the ground. The arrow hits the target after a total of 2 seconds at a height of 5 feet above the ground. Find the particular equation that models this data.

Answers

1. $(3, 1, -4)$, these three planes intersect in a point.

2. $(1, -5, -1)$, these three planes intersect in a point.

3. $(3, 1, 2.5)$, these three planes intersect in a point.

4. No solution or inconsistent. Two of these planes are parallel.

5. $(36, 24, 12)$, these three planes intersect in a point.

6. No solution. All three of these equations represent the same plane.

7. $y = -3x^2 + 5x - 10$

8. $y = 3x^2 - 42x + 153$

9. The equation that fits this data is $y = 0.04x^2 - 3.6x + 100$. According to this model, an 80-year old person would make 68 misunderstandings in a 24 hour period. A one year old would make approximately 96. The age that understands the most would be the age at which the number of misinterpretations is the lowest. This is at the vertex of this function. The vertex is at $(45, 19)$ so 45 year olds have the lowest number with only 19 mistakes.

10. With rounding, $y = -0.31x^2 + 0.43x + 5.4$.

Students turn their attention back to logarithms. Using Guess and Check, Pattern Recognition, and other problem solving strategies, students develop several properties of logs that enable them to solve equations that have been, until now, very cumbersome to solve. These properties are listed in the Math Notes box in Lesson 6.2.2.

Example 1

Solve each of the following equations for x.

a. $5^x = 67$

b. $3(7^x) + 4 = 124$

Each of these problems has the variable as the exponent, which makes them different from others that students have been solving. So far, students have been solving problems similar to these by Guess and Check. This approach has been time consuming and difficult to find an accurate answer.

Now students can use the log property, $\log(b^x) = x \log(b)$, to solve these equations for x. As with other equations, however, students must isolate the variable on one side of the equation. Note: The decimal answer is an approximation. The exact answer is the fraction $\frac{\log 67}{\log 5}$.

$$5^x = 67$$
$$\log(5^x) = \log(67)$$
$$x \log(5) = \log(67)$$
$$x = \frac{\log 67}{\log 5}$$
$$x \approx 2.61252$$

Some work must be done to the second equation before we can incorporate logs. We will move everything we can to one side of the equation so that the variable is as isolated as possible (steps 1 through 3).

$$3(7^x) + 4 = 124$$
$$3(7^x) = 120$$
$$7^x = 40$$
$$\log(7^x) = \log(40)$$
$$x \log(7) = \log(40)$$
$$x = \frac{\log 40}{\log 7}$$
$$x \approx 1.89571$$

Example 2

Using the properties of logs of products and quotients, rewrite each product as a sum, each quotient as a difference, and vice versa.

a. $\log_3(16x) =$

b. $\log_6(32) + \log_6(243) =$

c. $\log_8\left(\frac{3x}{7}\right) =$

d. $\log_{12}(276) - \log_{12}(23) =$

The two properties we will use are $\log(ab) = \log(a) + \log(b)$ and $\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$. These properties are true for any base, so we can use the first one to rewrite part (a) as $\log_3(16x) = \log_3(16) + \log_3(x)$. This new form is not necessarily better or simpler, it is just another way to represent the expression. In part (b), we can use the first property to write $\log_6(32) + \log_6(243) = \log_6(32 \cdot 243) = \log_6 7776$. Although it is not necessary, this can be simplified further. Since $6^5 = 7776$, $\log_6 7776 = 5$.

We will rewrite parts (c) and (d) using the second property listed above. Therefore, $\log_8\left(\frac{3x}{7}\right) = \log_8(3x) - \log_8(7)$. Note: We could use the first property to expand this further by writing $\log_8(3x)$ as $\log_8 3 + \log_8 x$. Working in the opposite direction on part (d), we write $\log_{12}(276) - \log_{12}(23) = \log_{12}\left(\frac{276}{23}\right)$. Simplifying further, $\log_{12}\left(\frac{276}{23}\right) = \log_{12} 12 = 1$.

Example 3

Fall came early in Piney Orchard, and the community swimming pool was still full when the first frost froze the leaves. The outside temperature hovered at 30°. Maintenance quickly turned off the heat so that energy would not be wasted heating a pool that nobody would be swimming in for at least six months. As Tess walked by the pool each day on her way to school, she would peer through the fence at the slowly cooling pool. She could just make out the thermometer across the deck that displayed the water's temperature. On the first day, she noted that the water temperature was 68°. Four days later, the temperature reading was 58°. Write an equation that models this data. If the outside temperature remains at 30°, and the pool is allowed to cool, how long before it freezes?

Heating and cooling problems are typical application problems that use exponential equations. In class, students solved such a problem, *The Case of the Cooling Corpse*. The equation that will model this problem is an exponential equation of the form $y = km^x + b$. In the problem description, we are given two data points: $(0, 68°)$ and $(4, 58°)$. We also have another piece of important information. The outside temperature is hovering at 30°. This is the temperature the water will approach, that is, $y = 30$ is the horizontal asymptote for this equation. Knowing this fact allows us to write the equation as $y = km^x + 30$. To determine k and m, we will substitute our values into the equation and solve for k and m.

$(0,68) \Rightarrow y = km^x + 30 \Rightarrow 68 = km^0 + 30$

$(4,58) \Rightarrow y = km^x + 30 \Rightarrow 58 = km^4 + 30$

This gives us two equations with two unknowns that we can solve. Simplifying first makes our work a lot easier. The first equation simplifies to $38 = k$ since $m^0 = 1$. Since $k = 38$ we can substitute this value into the second equation to determine m.

$$58 = km^4 + 30$$

$$58 = 38m^4 + 30$$

$$28 = 38m^4$$

$$m^4 = \tfrac{28}{38} \approx 0.7368$$

$$m \approx 0.9265$$

Therefore the equation is $y = 38(0.9265)^x + 30$. To determine when the pool will freeze, we want to find when the water's temperature reaches $32°$.

$$32 = 38(0.9265)^x + 30$$

$$2 = 38(0.9265)^x$$

$$\tfrac{2}{38} = 0.9265^x$$

$$\log\left(\tfrac{2}{38}\right) = \log 0.9265^x$$

$$\log\left(\tfrac{2}{38}\right) = x \log 0.9265$$

$$x = \frac{\log\left(\tfrac{2}{38}\right)}{\log 0.9265} \approx 38.57$$

In approximately 38 and a half days, the water in the pool will freeze if the outside temperature remains at $30°$ for those days. In reality, the pool would be drained to prevent damage from freezing.

Problems *answers follow*

all good practice

Solve each of the following equations for x.

1. $(2.3)^x = 7$

2. $12^x = 6$

3. $\log_7 49 = x$

4. $\log_3 x = 4$

5. $5(3.14)^x = 18$

6. $7x^8 = 294$

7. $\log_x 100 = 4$

8. $\log_5 45 = x$

9. $2(6.5)^x + 7 = 21$

10. $-\frac{1}{2}(14)^x + 6 = -9.1$

Rewrite each log of a product as a sum of logs, each difference of logs as a log of a quotient, and vice versa.

11. $\log(23 \cdot 3)$

12. $\log\left(\frac{3x}{8}\right)$

13. $\log_2\left(\frac{60}{7}\right)$

14. $\log_8 12 - \log_8 2$

15. $\log_5 25 + \log_5 25$

16. $\log(10 \cdot 10)$

17. $\log_{13}\left(15x^2\right)$

18. $\log 123 + \log 456$

19. $\log 10^8 - \log 10^3$

20. $\log(5x - 4)$

Simplify.

21. $\log_2 64$

22. $\log_{17} 17^{1/8}$

23. $8^{\log_8 1.3}$

24. $2.3^{5 \log_{2.3} 1}$

25. Climbing Mt. Everest is not an easy task! Not only is it a difficult hike, but the Earth's atmosphere decreases exponentially as you climb above the Earth's surface, and this makes it harder to breathe. The air pressure at the Earth's surface (sea level) is approximately 14.7 pounds per square inch (or 14.7 psi). In Denver, Colorado, elevation 5280 feet, the air pressure is approximately 12.15 psi. Write the particular equation representing this data expressing air pressure as a function of altitude. What is the air pressure in Mexico City, elevation 7300 feet? At the top of Mt. Everest, elevation 29,000 feet? (Note: You will need to carry out the decimal values several places to get an accurate equation and air pressures.)

Answers

1. $x = \frac{\log 7}{\log 2.3} \approx 2.336$

2. $x = \frac{\log 6}{\log 12} \approx 0.721$

3. $x = 2$

4. $x = 81$

5. $x = \frac{\log 3.6}{\log 3.14} \approx 1.119$

6. $x \approx 1.596$

7. $x \approx 3.162$

8. $x = \frac{\log 45}{\log 5} \approx 2.365$

9. $x = \frac{\log 7}{\log 6.5} \approx 1.040$

10. $x = \frac{\log 30.2}{\log 14} \approx 1.291$

11. $\log 23 + \log 3$

12. $\log (3x) - \log 8$

13. $\log_2 60 - \log_2 7$

14. $\log_8 \left(\frac{12}{2}\right) = \log_8 6$

15. $\log_5 625$

16. $\log 10 + \log 10$

17. $\log_{13} 15 + \log_{13} x^2$

18. $\log 56{,}088$

19. $\log \left(\frac{10^8}{10^3}\right) = \log 10^5$

20. Already simplified.

21. 6

22. $\frac{1}{8}$

23. 1.3

24. 1

25. The particular equation is $y = 14.7(0.999964)^x$ where x is the elevation, and y is the number of pounds per square inch (psi). The air pressure in Mexico City is approximately 11.3 psi, and at the top of Mt. Everest, the air pressure is approximately 5.175 psi.

9. If p and q are two different prime numbers greater than 2, and $n = pq$, how many positive factors, including 1 and n, does n have?

10. If $\frac{1}{2}\left(30x^2 + 20x^2 + 10x + 1\right) = ax^3 + bx^2 + cx + d$, for all values of x where a, b, c, and d are all constants, what is the value of $a + b + c + d$?

Answers

1. B
2. A
3. A
4. B
5. A
6. B
7. C
8. $r = 2.1$
9. 4
10. 30.5