Ch. 5 Extra practice with some topics

INVERSES 5.1.1 – 5.1.3

Students explore inverses, that is, equations that "undo" the actions of functions. Although they may not be aware of it, students have already seen inverses without calling them that. When solving equations, students reverse operations to undo the equation leaving just x, and that is what inverses do. Students further explore composition of functions by considering what happens when inverses are combined. For further information see the Math Notes boxes in Lessons 5.1.2 and 5.1.3.

Example 1

Find the inverse (undo) rule for the functions below. Use function notation and give the inverse rule a name different from the original function.

a. $f(x)=\frac{x-6}{3}$ b. $g(x)=(x+4)^2+1$

The function in part (a) subtracts six from the input then divides by three. The undo rule, or inverse, reverses this process. Therefore, the inverse first multiplies by three then adds six. If we call this inverse $h(x)$, we can write $h(x)=3x+6$.

The function $g(x)$ adds four to the input, squares that value, then adds one. The inverse will first subtract one, take the square root then subtract four. Calling this rule $j(x)$ we can write $j(x)=\pm\sqrt{x-1}-4$.

Rather than give the inverse a new name, we can use the notation for inverses. The inverse of $f(x)$ is written as $f^{-1}(x)$.

Note: The inverses $h(x)$ and $j(x)$, are fundamentally different. $h(x)=3x+6$ is the equation of a non-vertical line, therefore $h(x)$ is a function. $j(x)=\pm\sqrt{x-1}-4$, however, is not a function. By taking the square root, we created a positive value and a negative value. This gives two outputs for each input, and so by definition it is not a function.

Although we computed the inverses through a verbal description of what each function does, the students learn an algorithm for finding an inverse. They switch the x and y, and then solve for y. Using this algorithm on the equations above:

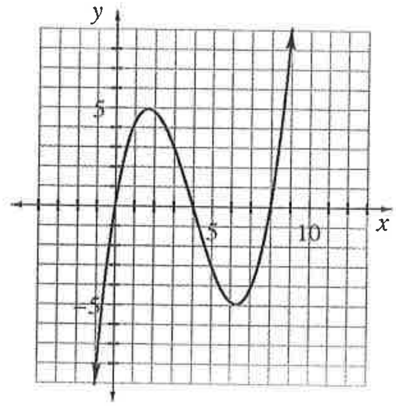
$$f(x)=\frac{x-6}{3}$$
$$y=\frac{x-6}{3}$$
$$x=\frac{y-6}{3}$$
$$3x=y-6$$
$$3x+6=y$$

$$g(x)=(x+4)^2+1$$
$$y=(x+4)^2+1$$
$$x=(y+4)^2+1$$
$$x-1=(y+4)^2$$
$$\pm\sqrt{x-1}=y+4$$
$$-4\pm\sqrt{x-1}=y$$

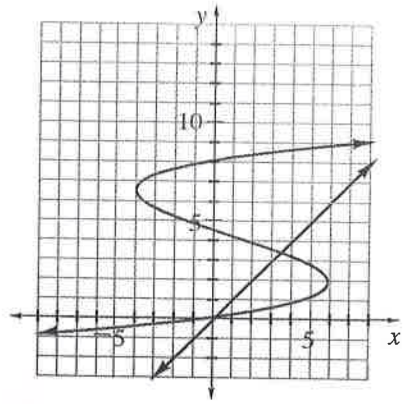
Example 2

The graph of $f(x) = 0.2x^3 - 2.4x^2 + 6.4x$ is shown at right. Graph the inverse of this function.

Following the algorithm for determining the equation for an inverse, as we did above, would be difficult here. The students do not have a method for solving cubic equations. Nevertheless, students can graph the inverse because they know a special property about the graphs of functions and their inverses: they are symmetrical about the line $y = x$.



If we add the line $y = x$ to the graph, the inverse is the reflection across this line. Here we can fold the paper along the line $y = x$, and trace the result to create the reflection.



SKIP

Example 3

Consider the function $f(x) = \frac{2x-1}{7}$. Determine the inverse of $f(x)$ and label it $g(x)$. Verify that these two functions are inverses by calculating $f(g(x))$ and $g(f(x))$.

Using the algorithm, we can determine the inverse.

$$y = \frac{2x-1}{7}$$
$$x = \frac{2y-1}{7}$$
$$7x = 2y - 1$$
$$7x + 1 = 2y$$
$$y = \frac{7x+1}{2}$$
$$g(x) = \frac{7x+1}{2}$$

Composing the two functions $f(x)$ and $g(x)$ gives a method for checking whether or not functions are inverses of each other. Since one function "undoes" the other, when the functions are composed, the output should be x.

$$f(x) = \frac{2x-1}{7} \qquad g(x) = \frac{7x+1}{2}$$

$$f(g(x)) = f\left(\frac{7x+1}{2}\right) = \frac{2\left(\frac{7x+1}{2}\right)-1}{7} = \frac{7x+1-1}{7} = \frac{7x}{7} = x$$

$$g(f(x)) = g\left(\frac{2x-1}{7}\right) = \frac{7\left(\frac{2x-1}{7}\right)+1}{2} = \frac{2x-1+1}{2} = \frac{2x}{2} = x$$

Since $f(g(x)) = g(f(x)) = x$, the functions are inverses.

Problems

Find the inverse of each of the following functions.

1. $f(x) = 8(x - 13)$
2. $y = -\frac{3}{4}x + 6$
3. $y = \frac{5(x+2)}{3}$
4. $f(x) = x^2 + 6$
5. $f(x) = \frac{3}{x} + 6$
6. $g(x) = \frac{5}{x}$
7. $g(x) = (x+1)^2 - 3$
8. $y = (x+2)^3$
9. $y = 3 + \sqrt{x-4}$
10. $g(x) = 6x + 2$

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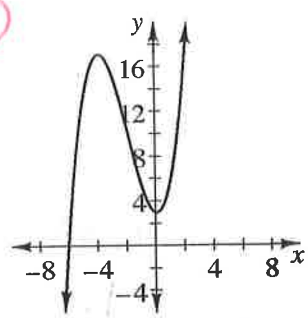
Sketch the graph of the inverse of each of the following functions.

11. $y = \frac{x}{6} + 2$
12. $f(x) = 2x^2 - 1$
13. $g(x) = x$
14. $y = \frac{1}{5x}$
15.



For each of the following pairs of functions, determine $f(g(x))$ and $g(f(x))$, then use the result to decide whether or not $f(x)$ and $g(x)$ are inverses of each other.

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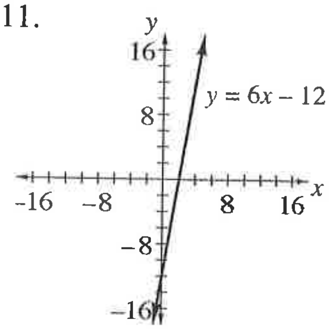
16. $f(x) = 5x + 7$
    $g(x) = \frac{x-7}{5}$
17. $f(x) = 8x$
    $g(x) = \frac{1}{8}x$
18. $f(x) = x + 5$
    $g(x) = \frac{1}{x+5}$
19. $f(x) = \frac{2}{3x}$
    $g(x) = \frac{3x}{2}$
20. $f(x) = \frac{2}{3}x + 6$
    $g(x) = \frac{3(x-6)}{2}$
21. $f(x) = x\sqrt{3} + 9$
    $g(x) = \left(\frac{x-9}{\sqrt{3}}\right)^2$

Answers

1. $y=\frac{x}{8}+13$
2. $y=-\frac{4}{3}x+8$
3. $y=\frac{3}{5}x-2$
4. $y=\pm\sqrt{x-6}$
5. $y=\frac{3}{x-6}$
6. $y=\frac{5}{x}$
7. $y=-1\pm\sqrt{x+3}$
8. $y=-2+\sqrt[3]{x}$
9. $y=(x-3)^2+4$, for $x\geq 3$
10. $y=\frac{x-2}{6}$

11.



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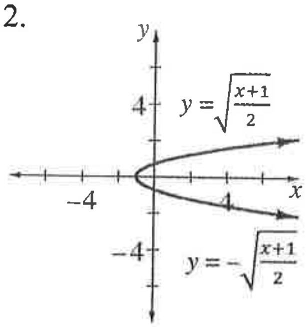

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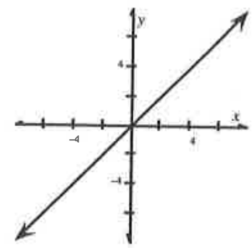

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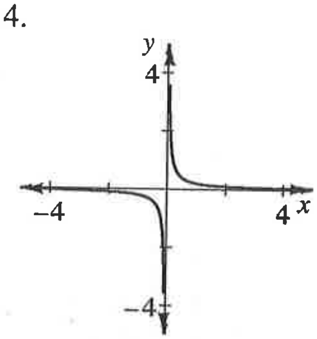

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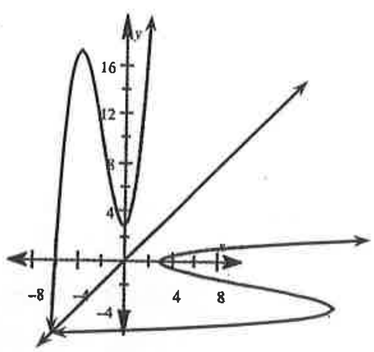


16. $f(g(x))=g(f(x))=x$. They are inverses.
17. $f(g(x))=g(f(x))=x$. They are inverses.
18. $f(g(x))=\frac{1}{x+5}+5$, $g(f(x))=\frac{1}{x+10}$. No, they are not inverses.
19. $f(g(x))=\frac{4}{9x}$, $g(f(x))=\frac{1}{x}$. No, they are not inverses.
20. $f(g(x))=g(f(x))=x$. They are inverses.
21. $f(g(x))=\frac{\sqrt{3}(x-9)^2}{3}+9$, $g(f(x))=x^2$. No, they are not inverses.

LOGARITHMS 5.2.1 – 5.2.4

The earlier sections of this chapter gave students many opportunities to find the inverses of various functions. Here, students explore the inverse of an exponential function. Although they can graph the inverse by reflecting the graph of an exponential function across the line $y = x$, they cannot write the equation of this new function. Writing the equation requires the introduction of a new function, the logarithm. Students explore the properties and graphs of logarithms, and in a later chapter use them to solve equations of this type. For further information see the Math Notes box in Lesson 5.2.2.

Example 1

Find each of the values below and then justify your answer by writing the equivalent exponential form.

a. $\log_5 25 = ?$ b. $\log_7 ? = 3$ c. $\log_2\left(\frac{1}{8}\right) = ?$

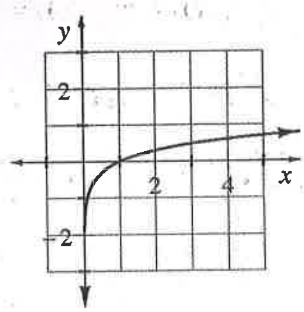
A logarithm is really just an exponent, so an expression like the one in part (a), $\log_5 25$, is asking "What exponent can I raise the base 5 to, to get 25?" We can translate this question into an equation: $5^? = 25$. By phrasing it this way, the answer is more apparent: 2. This is true because $5^2 = 25$.

Part (b) can be rephrased as $7^3 = ?$. The answer is 343.

Part (c) asks "2 to what exponent gives $\frac{1}{8}$?" or $2^? = \frac{1}{8}$. The answer is −3 because $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$.

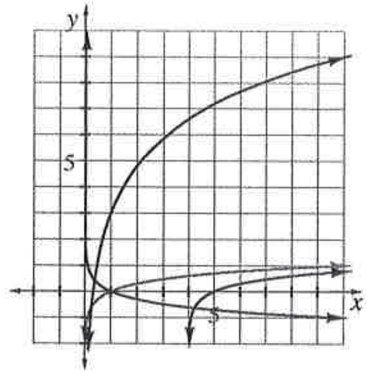
Example 2

The graph of $y = \log x$ is shown at right. Use this "parent graph" to graph each of the following equations. Explain how you get your new graphs.



$y = \log(x-4)$ $y = 6\log(x) + 3$ $y = -\log x$

The logarithm function follows the same rules for transforming its graphs as other functions we have used. The first equation shifts the original graph to the right four units. The graph of the second equation is shifted up three units (because of the "+ 3") but is also stretched because it is multiplied by six. The third function is flipped across the x-axis. All three of these graphs are shown at right. The original function $y = \log x$ is also there, in light gray. Note: When a logarithm is written without a base, as in $y = \log x$ and the log key used on a calculator, the base is 10.



Problems

(answers follow)

Rewrite each logarithmic equation as an exponential equation and vice versa.

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1. $y = \log_4 x$
2. $3 = \log_2 x$
3. $x = \log_5 30$
4. $4^x = 80$
5. $\left(\frac{1}{2}\right)^x = 64$
6. $x^3 = 343$
7. $5^x = \frac{1}{125}$
8. $\log_x 32 = y$
9. $11^3 = x$
10. $-4 = \log_x\left(\frac{1}{16}\right)$

What is the value of x in each equation below? If necessary, rewrite the expression in the equivalent exponential equation to verify your answer.

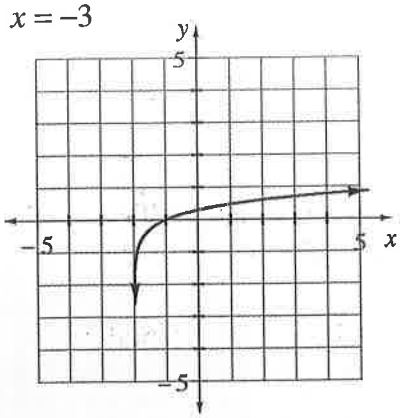
all

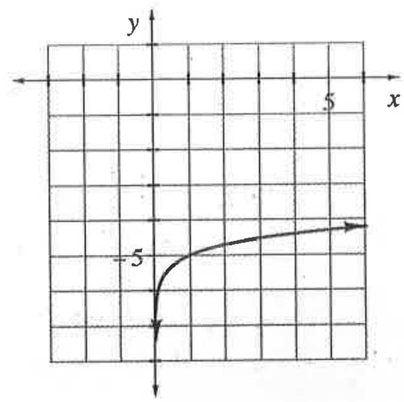
11. $4 = \log_5 x$
12. $2 = \log_9 x$
13. $9 = \log x$
14. $81 = 9^x$
15. $\left(\frac{1}{3}\right)^x = 243$
16. $6^x = 7776$
17. $7^x = \frac{1}{49}$
18. $\log_2 32 = x$
19. $\log_{11} x = 3$
20. $\log_5\left(\frac{1}{125}\right) = x$

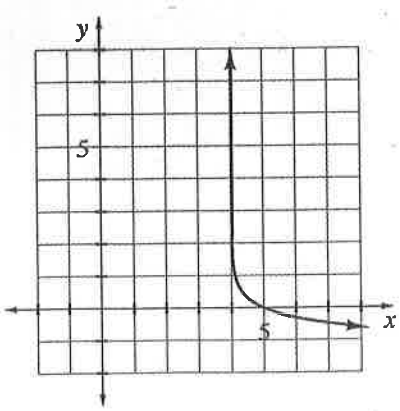
Graph each of the following equations.

21. $y = \log(x + 2)$
22. $y = -5 + \log x$
23. $y = -\log(x - 4)$
24. $y = 5 + 3\log(x - 7)$

Answers

1. $4^y = x$
2. $2^3 = x$
3. $5^x = 30$
4. $\log_4 80 = x$
5. $\log_{1/2} 64 = x$
6. $\log_x 343 = 3$
7. $\log_5 \left(\frac{1}{125}\right) = x$
8. $x^y = 32$
9. $\log_{11} x = 3$
10. $x^{-4} = \frac{1}{16}$
11. $x = 625$
12. $x = 81$
13. $x = 1{,}000{,}000{,}000$
14. $x = 2$
15. $x = -5$
16. $x = 5$
17. $x = -2$
18. $x = 5$
19. $x = 1{,}331$
20. $x = -3$
21. 

22. 

23. 

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