Ch. 4 Extra Practice with Some Topics

SOLVING SYSTEMS OF EQUATIONS 4.1.1 – 4.1.4

Students have been solving equations even before Algebra 1. Now they focus on what a solution means, both algebraically and graphically. By understanding the nature of solutions, students are able to solve equations in new and different ways. Their understanding also provides opportunities to solve some challenging applications. In this section they will extend their knowledge about solving one and two variable equations to solve systems with three variables.

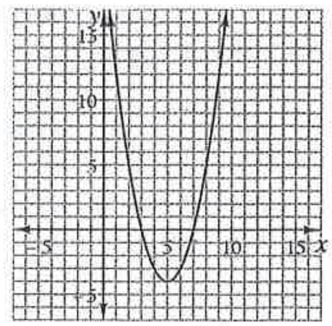
Example 1

The graph of $y=(x-5)^2-4$ is shown below right. Solve each of the following equations. Explain how the graph can be used to solve the equations.

a. $(x-5)^2-4=12$

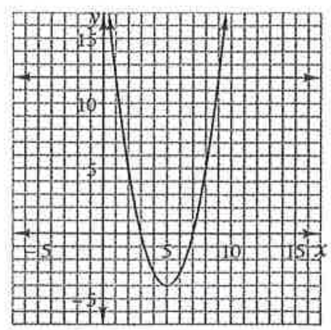
b. $(x-5)^2-4=-3$

c. $(x-5)^2=4$



Students can determine the correct answers through a variety of different ways. For part (a), most students would do the following:

$$(x-5)^2-4=12$$
$$(x-5)^2=16$$
$$x-5=\pm4$$
$$x=5\pm4$$
$$x=9,\ 1$$



This is correct and a standard procedure. However, with the graph of the parabola provided, the student can find the solution by inspecting the graph. Since we already have the graph of $y=(x-5)^2-4$, we can add the graph of $y = 12$ which is a horizontal line. These two graphs cross at two points, and the x-coordinates of these points are the solutions. The intersection points are (1, 12) and (9, 12). Therefore the solutions to the equation are $x = 1$ and $x = 9$.

We can use this method for part (b) as well. Draw the graph of $y = -3$ to find that the graphs intersect at (4, –3) and (6, –3). Therefore the solutions to part (b) are $x = 4$ and $x = 6$.

The equation in part (c) might look as if we cannot solve it with the graph, but we can. Granted, this is easy to solve algebraically:

$$(x-5)^2=4$$
$$x-5=\pm2$$
$$x=7,\ 3$$

But, we want students to be looking for alternative approaches to solving problems. By looking for and exploring alternative solutions, students are expanding their repertoire for solving problems. This year they will encounter equations that can only be solved through alternative methods. By recognizing the equation in part (c) as equivalent to $(x-5)^2-4=0$ (subtract four from both sides), we can use the graph to find where the parabola crosses the line $y = 0$ (the x-axis). The graph tells us the solutions are $x = 7$ and $x = 3$.

We substitute this value for x back into either one of the original equations to determine the value of y. Finally, we must check that the solution satisfies both equations.

$$y=-\frac{2}{5}x+3,\ x=5$$
$$y=-\frac{2}{5}(5)+3$$
$$y=-2+3$$
$$y=1$$

Solution to check: (5, 1)

Check

$$y=-\frac{2}{5}x+3$$
$$1\overset{?}{=}-\frac{2}{5}(5)+3$$
$$1=-2+3 \quad \text{Check}$$

$$y=\frac{3}{5}x-2$$
$$1\overset{?}{=}\frac{3}{5}(5)-2$$
$$1=3-2 \quad \text{Check}$$

Therefore the solution is the point (5, 1), which means that the graphs of these two lines intersect in one point, the point (5, 1).

(b) The equations in part (b) are written in the same form so we will solve this system the same way we did in part (a).

$$y=y$$
$$-2(x-2)^2+35=-2x+15$$
$$-2(x-2)^2=-2x-20$$
$$-2(x^2-4x+4)=-2x-20$$
$$-2x^2+8x-8=-2x-20$$
$$-2x^2+10x+12=0$$
$$x^2-5x-6=0$$
$$(x-6)(x+1)=0$$
$$x=6,\ x=-1$$

We substitute each x-value into either equation to find the corresponding y-value. Here we will use the simpler equation.

$$x=6,\ y=-2x+15$$
$$y=-2(6)+15$$
$$y=-12+15$$
$$y=3$$

Solution: (6, 3)

$$x=-1,\ y=-2x+15$$
$$y=-2(-1)+15$$
$$y=2+15$$
$$y=17$$

Solution: (−1, 17)

Lastly, we need to check each point in both equations to make sure we do not have any extraneous solutions.

$(6,3)$: $y=-2(x-2)^2+35$
$$3=-2(6-2)^2+35$$
$$3=-2(16)+35 \quad \text{Check}$$

$(6,3)$: $y=-2x+15$
$$3=-2(6)+15 \quad \text{Check}$$

$(-1,17)$: $y=-2(x-2)^2+35$
$$17=-2(-1-2)^2+35$$
$$17=-2(9)+35 \quad \text{Check}$$

$(-1,17)$: $y=-2x+15$
$$17=-2(-1)+15 \quad \text{Check}$$

In solving these two equations with two unknowns, we found two solutions, both of which check in the original equations. This means that the graphs of the equations, a parabola and a line, intersect in exactly two distinct points.

Example 4

Jo has small containers of lemonade and lime soda. She once mixed one lemonade container with three containers of lime soda to make 17 ounces of a tasty drink. Another time, she combined five containers of lemonade with six containers of lime soda to produce 58 ounces of another splendid beverage. Given this information, how many ounces are in each small container of lemonade and lime soda?

We can solve this problem by using a system of equations. To start, we let x equal the number of ounces of lemonade in each small container, and y equal the number of ounces of lime soda in each of its small containers. We can write an equation that describes each mixture Jo created. The first mixture used one ounce of x liquid and three ounces of y liquid to produce 17 ounces. This can be represented as $1x+3y=17$. The second mixture used five ounces of x liquid and six ounces of y liquid to produce 58 ounces. This can be represented by the equation $5x+6y=58$. We can solve this system to find the values for x and y.

$$\begin{aligned} x+3y=17 &\quad \xrightarrow{\times 5} \quad 5x+15y=85 \\ 5x+6y=58 &\quad \rightarrow \quad \underline{5x+6y=58} \\ &\qquad\qquad 9y=27 \\ &\qquad\qquad y=3 \end{aligned}$$

$$\begin{aligned} x+3(3)&=17 \\ x+9&=17 \\ x&=8 \end{aligned}$$

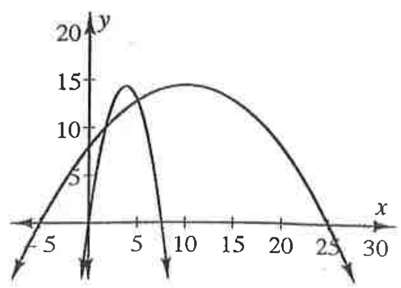
(Note: you should check these values!) Therefore each container of Jo's lemonade has 8 ounces, and each container of her lime soda has only 3 ounces.

20. The next math test will contain 50 questions. Some will be worth three points while the rest will be worth six points. If the test is worth 195 points, how many three-point questions are there, and how many six-point questions are there?

21. Reread Example 3 from Chapter 4 about Dudley's water balloon fight. If you did this problem, you found that Dudley's water balloons followed the path described by the equation $y = -\frac{8}{125}(x-10)^2 + \frac{72}{5}$. Suppose Dudley's nemesis, in a mad dash to save his base from total water balloon bombardment, ran to the wall and set up his launcher at its base. Dudley's nemesis launches his balloons to follow the path $y = -x\left(x - \frac{189}{25}\right)$ in an effort to knock Dudley's water bombs out of the air. Is Dudley's nemesis successful? Explain.

Answers

1. $(4, 7)$
2. $(-2, 5)$
3. no solution
4. $\left(-\frac{1}{2}, 11\right)$
5. All real numbers
6. $(12, 3)$
7. $x = 4$. The horizontal line $y = 3$ crosses the parabola in only one point, at the vertex.
8. $x = 2, x = 6$
9. No solution. The horizontal line $y = 1$ does not cross the parabola.
10. $x = 0, x = 8$. Add three to both sides to rewrite the equation as $\frac{1}{2}(x-4)^2 + 3 = 11$. The horizontal line $y = 11$ crosses at these two points.
11. $x = 7, x = 1$
12. no solution
13. $x = 2$
14. No solution. (A square root must have a positive result.)
15. All real numbers. When graphed, these equations give the same line.
16. $(0, 4)$. The parabola and the line intersect only once.
17. No solution. This parabola and this line do not intersect.
18. $(2, -2)$ and $(5, -5)$. The line and the parabola intersect twice.
19. 145 adult tickets were sold, while 290 child tickets were sold.
20. There are 35 three-point questions and 15 six-point questions on the test.
21. By graphing we see that the nemesis' balloon when launched at the base of the wall (the y-axis), hits the path of the Dudley's water balloon. Therefore, if timed correctly, the nemesis is successful.



Next we choose a point in the middle section. An easy value to try is $x = 0$.

$$(0)^2 - 4(0) - 5 \overset{?}{<} 0$$
$$0 - 0 - 5 \overset{?}{<} 0$$
$$-5 \overset{?}{<} 0$$

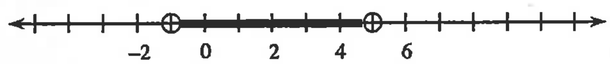
True! This section is part of the solution.

Finally, we check to see if any points in the last section make the inequality true. Try $x = 7$.

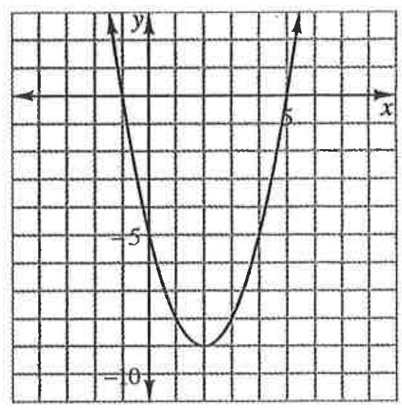
$$(7)^2 - 4(7) - 5 \overset{?}{<} 0$$
$$49 - 28 - 5 \overset{?}{<} 0$$
$$16 \overset{?}{<} 0$$

False! Therefore the solution is only the middle section, the numbers that lie between -1 and 5.

We can represent this in a couple of ways. We can use symbols to write $-1 < x < 5$. We can also represent the solution on the number line by shading the section of the number line that represents the solution of the inequality. Any point in the shaded section of the number line will make the inequality true.

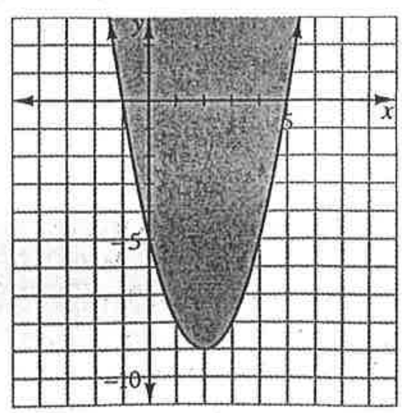


The last inequality of the example has added a y. We want to find all y-values greater than or equal to the quadratic expression. Having both x and y means we need to use an xy-coordinate graph. The graph of the parabola at right divides the plane into two regions: the part within the "bowl" of the parabola – the interior – and the region outside the parabola. The points <u>on</u> the parabola represent where $y = x^2 - 4x - 5$. We use a test point from one of the regions to check whether it will make the inequality true or false. As before, we are looking for the "true" region. The point (0, 0) is an easy point to use.



$$0 \overset{?}{\geq} (0)^2 - 4(0) - 5$$
$$0 \overset{?}{\geq} 0 - 0 - 5$$
$$0 \overset{?}{\geq} -5$$

True! Therefore the region containing the point (0, 0) is the solution. This means any point chosen in this region, the "bowl" of the parabola, will make the inequality true.



To illustrate that this region is the solution, we shade this region of the graph. Note: Since the inequality was "greater than **or equal to**," the parabola itself is included in the solution. If the inequality had been strictly "greater than," we would have made the curve dashed to illustrate that the parabola itself is not part of the solution.

To see how these equations and inequalities are related, examine the graph of the parabola. Where are the y-values of the parabola negative? Where are they equal to zero? The graph is negative when it dips below the x-axis, and this happens when x is between -1 and 5. Solving the first inequality told you that as well. It equals zero at the points $x = -1$ and $x = 5$, which you found by solving the equation. Therefore, if we had the graph initially, we could have answered the first two parts quickly by looking at the graph.