EQUIVALENCE

3.1.1 - 3.1.3

Solving equations is a skill that algebra students practice a great deal. In Algebra 2, the equations become increasingly more complex. Whenever possible, it is beneficial for students to rewrite equations in a simpler form, or as equations they already know how to solve. This is done by recognizing equivalent expressions and developing algebraic strategies for demonstrating equivalence.

Extra Practice

Example 1 10

Emma and Rueben have been given a sequence they have never seen before. It does not seem to be an arithmetic or a geometric sequence.

n	t(n)
0	-30
1	-36
2	-40
3	-42
4	-42
5	-40
6	-36

After a great deal of brain strain, Emma exclaims "Hey! I see a pattern! If we look at the differences between the t(n)'s, we can list the whole sequence!" Rueben agrees, "Oh, I see it! But what a pain to list it all out. We should be able to find a formula."

"Now that we see a pattern," Emma says, "Let's each spend some time thinking of a formula."

After a few minutes, both Rueben and Emma have formulas. "Wait a minute!" says Rueben. "That's not the formula I got. My formula is $t(n) = n^2 - 7n - 30$, but your formula is t(n) = (n+3)(n-10). Which one of us is correct?"

a. Who is correct? Justify your answer completely.

b. Later Tess, another team member, says "Ha! I have the right equation! It is

 $t(n) = \left(n - \frac{7}{2}\right)^2 - \frac{169}{4}$." Rueben comments "You are really off, Tess. That is nowhere near the right answer!" Is Rueben correct, or has Tess found another equation? Justify your answer.

We can show that both Rueben and Emma's equations produce the values in the table by substituting different values for n, but that would only show that they are equivalent for those specific values. We must show that the two equations are equivalent algebraically in order to verify that they are the same. To do this, we can use algebra to rewrite one equation, and hopefully get the other one.

$$t(n) = (n+3)(n-10)$$

= $n^2 - 10n + 3n - 30$
= $n^2 - 7n - 30$

We started with Rueben's equation, and through algebraic manipulation, the result was Emma's equation. Therefore the equations are equivalent.

Parent Guide and Extra Practice

Similarly, we can manipulate Tess' equation to see if we can get either of the other two equations. If we can, then it too is equivalent. Start by expanding $\left(n-\frac{7}{2}\right)^2$.

$$t(n) = \left(n - \frac{7}{2}\right)^2 - \frac{169}{4}$$

= $n^2 - 2\left(\frac{7}{2}\right)(n) + \frac{49}{4} - \frac{169}{4}$
= $n^2 - 7n - \frac{120}{4}$
= $n^2 - 7n - 30$

Tess has an equivalent equation as well. Therefore all three are equivalent equations.

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Example 2

Solve the following equation by rewriting it as a simpler equivalent equation: $\frac{1}{32}x^2 - \frac{1}{8}x = 1$.

This equation would be much simpler if it did not have any fractions, so multiply everything by 32 to eliminate the denominators.

 $\frac{\frac{1}{32}x^2 - \frac{1}{8}x = 1}{32\left(\frac{1}{32}x^2 - \frac{1}{8}x\right)^2 = 32(1)}$ it's better, in my opinion, to write 32 next to each term rather than 1) $32\left(\frac{1}{32}x^2\right) - 32\left(\frac{1}{8}x\right) = 32$ $x^2 - 4x = 32$

To solve a quadratic equation, set it equal to zero, and solve by either factoring or using the Quadratic Formula. Since the equation appears to be easily factorable, use that method.

$$x^{2} - 4x = 32$$

$$x^{2} - 4x - 32 = 0$$

$$(x - 8)(x + 4) = 0$$

$$x - 8 = 0, x + 4 = 0$$

$$x = 8, x = -4$$

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Chapter 3

Example 3

Decide whether or not the following pairs of expressions or equations are equivalent for all values of the variables. Justify your answer completely.

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a. $\sqrt{a+b}$ and $\sqrt{a} + \sqrt{b}$ b. $\frac{12}{x+4}$ and $\frac{12}{x} + \frac{12}{4}$ c. $\frac{x+4}{12}$ and $\frac{x}{12} + \frac{4}{12}$ d. $3x^2 + 6x - 1 = x^2 + 18x - 14$ and $2x^2 - 12x + 13 = 0$

In part (a), choose different values for a and b to check. For instance, if a = 1 and b = 2, then we would have $\sqrt{1+2} = \sqrt{3} \approx 1.732$, and $\sqrt{1} + \sqrt{2} = 1 + \sqrt{2} \approx 2.414$. Therefore the two expressions are not equal. (Note: These expressions are only equal when both a and b are equal to zero.)

Try any value for x in part (b), and the two expressions will not be equal. For example, if x = 1, then $\frac{12}{x+4} = \frac{12}{1+4} = \frac{12}{5}$ and $\frac{12}{x} + \frac{12}{4} = \frac{12}{1} + \frac{12}{4} = 12 + 3 = 15$.

Note that you only need to find *one* example the does not work to demonstrate that the two expressions of equations are not equivalent. This strategy is known as a counterexample.

The expressions in part (c) demonstrate how we add fractions with common denominators: by adding the numerators.

You may wish to try some values of x in the two equations of part (d), but the equations are fairly messy. In addition, using a few values would not show that the equations are equivalent for *all* values of x. It is easier to simplify the first equation to see if it results in the second equation.

$$3x^{2} + 6x - 1 = x^{2} + 18x - 14$$
$$2x^{2} + 6x - 1 = 18x - 14$$
$$2x^{2} - 12x - 1 = -14$$
$$2x^{2} - 12x + 13 = 0$$

The result is the second equation. Therefore, these two equations are equivalent.

Rewrite the following expressions in a simpler form.

Problems

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1.
$$(3x^{-2})^{-5}(4x^3)$$
 2. $\frac{3x^{-4}y^3}{x^3y^{-5}}$ 3. $\left|\frac{-27x^{-12}y^8z}{(-3x^4y^2z)^{-2}}\right|^5$

Decide whether or not the following pairs of expressions or equations are equivalent for any values of the variables. Justify your answer completely.

For each sequence below there are two equations. Decide whether or not the equations represent the sequence, and whether or not the equations are equivalent. Justify your answer.

Simplify, and then solve the following equations.

12.
$$100x^2 + 500x = -600$$

13. $4x + 2y = 30$
 $2x - y = 5$
14. $\frac{x-1}{11} - \frac{7}{66} = \frac{1}{6}$
15. $\frac{x^2 + 3x + 2}{x^2 - x - 6} + \frac{x^2 + 4x - 5}{x^2 + 2x - 15} = \frac{x^2 + 6x}{x^2 + 4x - 12}$

-0

Answers

- 1. $\frac{4x^{13}}{243}$
- 2. $\frac{3y^8}{x^7}$
- 3. 1
- 4. No, in the first equation, x = -1. The second equation has no solution.
- 5. No, (0, 3) works in the first equation but not in the second. The standard form of the second equation is 3x 4y = 12.
- 6. Equivalent. Multiply the two binomials in the first equation go get $0.25x^2 0.5x 2 = 0$. Then multiply all of the terms by 8.
- 7. No, rewriting the equations in slope-intercept form produces different y-intercepts.
- 8. A and B both represent the sequence, and are equivalent.
- 9. A and B both represent the sequence, and are equivalent.
- 10. A and B both represent the sequence, and are equivalent.
- 11. Only B represents the sequence. They are not equivalent.
- 12. $x^2 + 5x + 6 = 0, x = -2, x = -3$
- 13. (5,5)
- 14. x = 4
- 15. x = 0, 1

SIMPLIFYING EXPRESSIONS

To **simplify rational expressions**, find factors in the numerator and denominator that are the same and then write them as fractions equal to 1. For example,

$\frac{6}{6} = 1$	x^2	$\frac{(x+2)}{x+2} = 1$	(3x-2)
$\frac{-}{6} = 1$	$\frac{x}{x^2} = 1$	$\frac{x}{(x+2)} = 1$	$\frac{(3x-2)}{(3x-2)} = 1$

Notice that the last two examples involved binomial sums and differences. Only when sums or differences are exactly the same does the fraction equal 1. The rational expressions below cannot be simplified:

$$\frac{(6+5)}{6}$$
 $\frac{x^3+y}{x^3}$ $\frac{x}{x+2}$ $\frac{3x-2}{2}$

As shown in the examples below, most problems that involve simplifying rational expressions will require that you **factor** the numerator and denominator.

Note that in all cases we assume the denominator does not equal zero, so in example 4 below the simplification is only valid provided $x \neq -6$ or 2. For more information, see examples 1 and 2 in the Math Notes box in Lesson 3.2.4.

One other special situation is shown in the following examples:

$$\frac{-2}{2} = -1 \qquad \frac{-x}{x} = -1 \qquad \frac{-x-2}{x+2} \Rightarrow \frac{-(x+2)}{x+2} \Rightarrow -1 \qquad \frac{5-x}{x-5} \Rightarrow \frac{-(x-5)}{x-5} \Rightarrow -1$$

Again assume the denominator does not equal zero.

Example 1

$$\frac{12}{54} = \frac{2 \cdot 2 \cdot 3}{2 \cdot 3 \cdot 3 \cdot 3} = \frac{2}{9} \text{ since } \frac{2}{2} = \frac{3}{3} = 1$$

$$\frac{6x^3y^2}{15x^2y^4} = \frac{2 \cdot 3 \cdot x^2 \cdot x \cdot y^2}{5 \cdot 3 \cdot x^2 \cdot y^2 \cdot y^2} = \frac{2x}{5y^2}$$
Example 3

$$\frac{12(x-1)^3(x+2)}{3(x-1)^2(x+2)^2} = \frac{4 \cdot 3(x-1)^2(x-1)(x+2)}{3(x-1)^2(x+2)(x+2)}$$

$$\frac{x^2 - 6x + 8}{x^2 + 4x - 12} = \frac{(x-4)(x-2)}{(x+6)(x-2)}$$

$$= \frac{4(x-1)}{(x+2)} \text{ since } \frac{3}{3}, \frac{(x-1)^2}{(x-1)^2}, \text{ and } \frac{x+2}{x+2} = 1$$

$$= \frac{(x-4)}{(x+6)} \text{ since } \frac{(x-2)}{(x-2)} = 1$$

3.2.2

Problems and wer

Simplify each of the following expressions completely. Assume the denominator does not equal zero.

Parent Guide and Extra Practice

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MULTIPLICATION AND DIVISION **OF RATIONAL EXPRESSIONS**

Multiplication or division of rational expressions follows the same procedure used with numerical fractions. However, it is often necessary to factor the polynomials in order to simplify them. When dividing rational expressions, change the problem to multiplication by inverting (flipping) the second expression (or any expression that follows a division sign) and completing the process as you do for multiplication. As in the previous section, remember that simplification assumes that the denominator is not equal to zero. For addition information, see examples 3 and 4 in the Math Notes box in Lesson 3.2.4.

Example 1

Multiply $\frac{x^2+6x}{(x+6)^2} \cdot \frac{x^2+7x+6}{x^2-1}$ and simplify the result.

After factoring, the expression becomes:

After multiplying, reorder the factors:

Since
$$\frac{(x+6)}{(x+6)} = 1$$
 and $\frac{(x+1)}{(x+1)} = 1$, simplify:

 $\frac{(x+6)}{(x+6)} \cdot \frac{(x+6)}{(x+6)} \cdot \frac{x}{(x-1)} \cdot \frac{(x+1)}{(x+1)}$ $1 \cdot 1 \cdot \frac{x}{x-1} \cdot 1 \implies \frac{x}{x-1}$ for $x \neq 6, -1, or 1$.

 $\frac{x(x+6)}{(x+6)(x+6)} \cdot \frac{(x+6)(x+1)}{(x+1)(x-1)}$

Example 2 Divide $\frac{x^2 - 4x - 5}{x^2 - 4x + 4} \div \frac{x^2 - 2x - 15}{x^2 + 4x - 12}$ and simplify the result.

First change to a multiplication expression by inverting (flipping) the second fraction:

After factoring, the expression is:

Reorder the factors (if you need to):

Since
$$\frac{(x-5)}{(x-5)} = 1$$
 and $\frac{(x-2)}{(x-2)} = 1$, simplify: $\frac{(x+1)}{(x-2)} \cdot \frac{(x+1)}{(x+1)}$

Thus, $\frac{x^2 - 4x - 5}{x^2 - 4x + 4} \div \frac{x^2 - 2x - 15}{x^2 + 4x - 12} = \frac{(x+1)}{(x-2)} \cdot \frac{(x+6)}{(x+3)}$ or $\frac{x^2 + 7x + 6}{x^2 + x - 6}$ for $x \neq -3, 2, \text{ or } 5$.

3.2.3

$$\frac{x^2 - 4x - 5}{x^2 - 4x + 4} \cdot \frac{x^2 + 4x - 12}{x^2 - 2x - 15}$$
$$\frac{(x - 5)(x + 1)}{(x - 2)(x - 2)} \cdot \frac{(x + 6)(x - 2)}{(x - 5)(x + 3)}$$

$$\frac{(x-5)}{(x-5)} \cdot \frac{(x-2)}{(x-2)} \cdot \frac{(x+1)}{(x-2)} \cdot \frac{(x+6)}{(x+3)}$$

$$\frac{(x+1)}{(x-2)} \cdot \frac{(x+6)}{(x+3)}$$

Problems 1.1

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Multiply or divide each pair of rational expressions. Simplify the result. Assume the denominator is not equal to zero.

1.
$$\frac{x^{2} + 5x + 6}{x^{2} - 4x} \cdot \frac{4x}{x + 2}$$
3.
$$\frac{x^{2} - 16}{(x - 4)^{2}} \cdot \frac{x^{2} - 3x - 18}{x^{2} - 2x - 24}$$
5.
$$\frac{x^{2} - x - 6}{x^{2} - x - 20} \cdot \frac{x^{2} + 6x + 8}{x^{2} - x - 6}$$
7.
$$\frac{15 - 5x}{x^{2} - x - 6} \div \frac{5x}{x^{2} + 6x + 8}$$
9.
$$\frac{2x^{2} - 5x - 3}{3x^{2} - 10x + 3} \cdot \frac{9x^{2} - 1}{4x^{2} + 4x + 1}$$
11.
$$\frac{3x - 21}{x^{2} - 49} \div \frac{3x}{x^{2} + 7x}$$
13.
$$\frac{y^{2} - y}{w^{2} - y^{2}} \div \frac{y^{2} - 2y + 1}{1 - y}$$
15.
$$\frac{x^{2} + 7x + 10}{x + 2} \div \frac{x^{2} + 2x - 15}{x + 2}$$

2.
$$\frac{x^2 - 2x}{x^2 - 4x + 4} \div \frac{4x^2}{x - 2}$$
4.
$$\frac{x^2 - x - 6}{x^2 + 3x - 10} \cdot \frac{x^2 + 2x - 15}{x^2 - 6x + 9}$$
6.
$$\frac{x^2 - x - 30}{x^2 + 13x + 40} \cdot \frac{x^2 + 11x + 24}{x^2 - 9x + 18}$$
8.
$$\frac{17x + 119}{x^2 + 5x - 14} \div \frac{9x - 1}{x^2 - 3x + 2}$$
10.
$$\frac{x^2 - 1}{x^2 - 6x - 7} \div \frac{x^3 + x^2 - 2x}{x - 7}$$
12.
$$\frac{x^2 - y^2}{x + y} \cdot \frac{1}{x - y}$$
14.
$$\frac{y^2 - y - 12}{y + 2} \div \frac{y - 4}{y^2 - 4y - 12}$$

Answers

1. $\frac{4(x+3)}{x-4}$	2. $\frac{1}{4x}$	3. $\frac{x+3}{x-4}$
$4. \ \frac{x+2}{x-2}$	$5. \ \frac{x+2}{x-5}$	6. $\frac{x+3}{x-3}$
7. $\frac{-x-4}{x}$	8. $\frac{17(x-1)}{9x-1}$	9. $\frac{3x+1}{2x+1}$
10. $\frac{1}{x(x+2)}$	11. 1	12. 1
$13. \qquad \frac{-y}{w^2 - y^2}$	14. $(y+3)(y-6)$	15. $\frac{x+2}{x-3}$

ADDITION AND SUBTRACTION OF RATIONAL EXPRESSIONS

Addition and Subtraction of Rational Expressions uses the same process as simple numerical fractions. First, find a common denominator (if necessary). Second, convert the original fractions to equivalent ones with the common denominator. Third, add (or subtract) the new numerators over the common denominator. Finally, factor the numerator and denominator and reduce (if possible). For additional information, see the Math Notes box in Lesson 3.2.5. Note that these steps are only valid provided that the denominator is not zero.

Example 1

The least common multiple of 2(n+2) and n(n+2) is 2n(n+2).

 $\frac{3}{2(n+2)} + \frac{3}{n(n+2)}$

To get a common denominator in the first fraction, multiply the fraction by $\frac{n}{n}$, a form of the number 1. Multiply the second fraction by $\frac{2}{2}$.

$$=\frac{3}{2(n+2)}\cdot\frac{n}{n}+\frac{3}{n(n+2)}\cdot\frac{2}{2}$$

Multiply the numerator and denominator of each term. It may be necessary to distribute the numerator.

Add, factor, and simplify the result. (Note: $n \neq 0$ or 2)

$$= \frac{3n}{2n(n+2)} + \frac{6}{2n(n+2)}$$
$$= \frac{3n+6}{2n(n+2)} \Rightarrow \frac{3(n+2)}{2n(n+2)} \Rightarrow \frac{3}{2n}$$

Example 2

$$\frac{2-x}{x+4} + \frac{3x+6}{x+4} \implies \frac{2-x+3x+6}{x+4} \implies \frac{2x+8}{x+4} \implies \frac{2(x+4)}{x+4} \implies 2$$
Example 3

$$\frac{3}{x-1} - \frac{2}{x-2} \implies \frac{3}{x-1} \cdot \frac{x-2}{x-2} - \frac{2}{x-2} \cdot \frac{x-1}{x-1} \implies \frac{3x-6-2x+2}{(x-1)(x-2)} \implies \frac{x-4}{(x-1)(x-2)}$$

Problems

Add or subtract each expression and simplify the result. In each case assume the denominator does not equal zero.

$$\begin{array}{rcl}
\hline 1. & \frac{x}{(x+2)(x+3)} + \frac{2}{(x+2)(x+3)} \\
\hline \hline (x+2)(x+3) + \frac{x-2}{(x+2)(x+3)} \\
\hline \hline (x+2)(x+3) + \frac{x-6}{b^2+2b-3} \\
\hline 5. & \frac{x+10}{x+2} + \frac{x-6}{x+2} \\
\hline 7. & \frac{3x-4}{3x+3} - \frac{2x-5}{3x+3} \\
\hline 9. & \frac{6a}{5a^2+a} - \frac{a-1}{5a^2+a} \\
\hline 11. & \frac{6}{5a^2+a} - \frac{a-1}{5a^2+a} \\
\hline 13. & \frac{5x+6}{x^2} - \frac{5}{x} \\
\hline 15. & \frac{10a}{a^2+6a} - \frac{3}{3a+18} \\
\hline 17. & \frac{5x+9}{x^2-2x-3} + \frac{6}{x^2-7x+12} \\
\hline 19. & \frac{3x+1}{x^2-16} - \frac{3x+5}{x^2+8x+16} \\
\hline\end{array}$$

2.
$$\frac{x}{x^{2}+6x+8} + \frac{4}{x^{2}+6x+8}$$
4.
$$\frac{2a}{a^{2}+2a+1} + \frac{2}{a^{2}+2a+1}$$
6.
$$\frac{a+2b}{a+b} + \frac{2a+b}{a+b}$$
8.
$$\frac{3x}{4x-12} - \frac{9}{4x-12}$$
10.
$$\frac{x^{2}+3x-5}{10} - \frac{x^{2}-2x+10}{10}$$
12.
$$\frac{5}{x-7} + \frac{3}{4(x-7)}$$
14.
$$\frac{2}{x+4} - \frac{x-4}{x^{2}-16}$$
16.
$$\frac{3x}{2x^{2}-8x} + \frac{2}{x-4}$$
18.
$$\frac{x+4}{x^{2}-3x-28} - \frac{x-5}{x^{2}+2x-35}$$
20.
$$\frac{7x-1}{x^{2}-2x-3} - \frac{6x}{x^{2}-x-2}$$

Answers

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1.
$$\frac{1}{x+3}$$
 2. $\frac{1}{x+2}$
5. 2 6. 3
9. $\frac{1}{a}$ 10. $\frac{x-3}{2}$
13. $\frac{6}{x^2}$ 14. $\frac{1}{x+4}$
17. $\frac{5(x+2)}{(x-4)(x+1)} = \frac{5x+10}{x^2-3x-4}$
19. $\frac{4(5x+6)}{(x-4)(x+4)^2}$

3.
$$\frac{b-3}{b-1}$$
 4. $\frac{2}{a+1}$
7. $\frac{1}{3}$ 8. $\frac{3}{4}$
11. $\frac{2}{x}$ 12. $\frac{23}{4(x-7)} = \frac{23}{4x-28}$
15. $\frac{9}{a+6}$ 16. $\frac{7}{2(x-4)} = \frac{7}{2x-8}$
18. $\frac{14}{(x-7)(x+7)} = \frac{14}{x^2-49}$
20. $\frac{x+2}{(x-3)(x-2)} = \frac{x+2}{x^2-5x+6}$

Parent Guide and Extra Practice