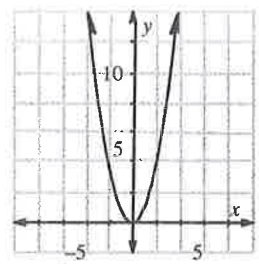
Ch. 2 Extra Practice

TRANSFORMATIONS OF $f(x) = x^2$ 2.1.1 – 2.1.5

In order for the students to be proficient in modeling data or contextual relationships, they must easily recognize and manipulate graphs of various functions. Students investigate the general equation for a family of quadratic functions, discovering ways to shift and change the graphs. Additionally, they learn how to quickly graph a quadratic function when it is written in graphing form. For further information see the Math Notes box in Lesson 2.1.4.

Example 1

The graph of $f(x) = x^2$ is shown at right. For each new function listed below, explain how the new graph would differ from this original graph.

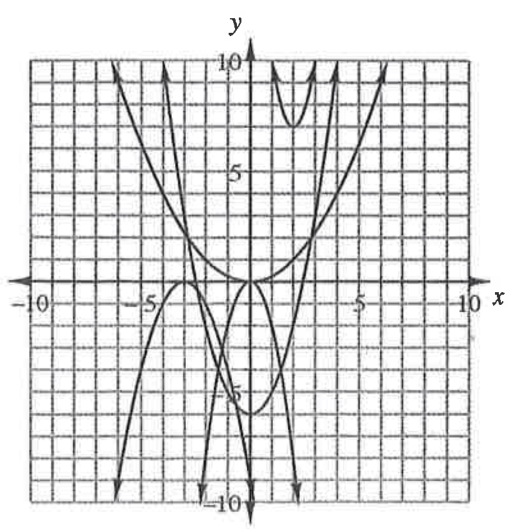


$g(x) = -2x^2$ $\quad$ $h(x) = (x+3)^2$ $\quad$ $j(x) = x^2 - 6$

$k(x) = \frac{1}{4}x^2$ $\quad$ $l(x) = 3(x-2)^2 + 7$

Every function listed above has something in common: they all have 2 as the highest power of x. This means that all of these functions are quadratic functions, and all will form a parabola when graphed. The only differences will be in the direction of opening (up or down), the size (compressed or stretched), and/or the location of the vertex.

The "-2" in $g(x) = -2x^2$ does two things to the parabola. The negative sign changes the parabola's direction so that it will open downward. The "2" stretches the graph making it appear skinnier. The graph of $h(x) = (x+3)^2$ will have the same shape as $f(x) = x^2$, open upward, and have a new location: it will move to the **left** three units. The graph of $j(x) = x^2 - 6$ will also have the same shape as $f(x) = x^2$, open upward and be shifted **down** six units.



The function $k(x) = \frac{1}{4}x^2$ does not move, still opens upward, but the $\frac{1}{4}$ will compress the parabola, making it appear "fatter." The last function, $l(x) = 3(x-2)^2 + 7$, combines all of these changes into one graph. The "3" causes the graph to be skinnier and open upward, the "–2" causes it to shift to the **right** two units, and the "+ 7" causes the graph to shift **up** 7 units. All these graphs are shown at right. Match the equation with the correct parabola.

Example 2

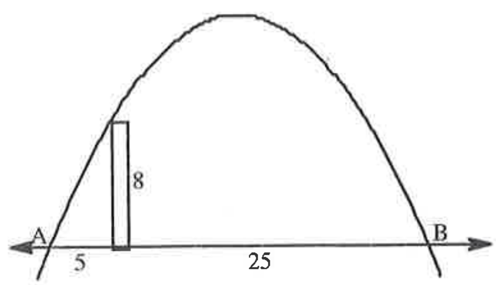
For each of the quadratic equations below, what is the vertex?

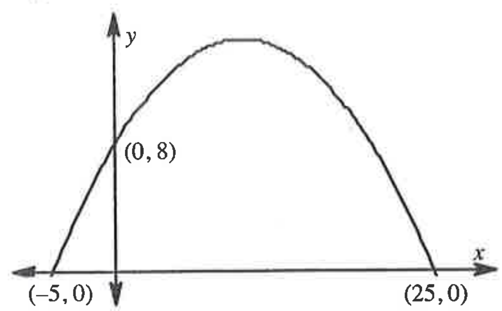
$f(x) = -2(x+4)^2 + 7$ $\qquad$ $g(x) = 5(x-8)^2$ $\qquad$ $h(x) = \frac{3}{5}x^2 - \frac{2}{5}$

For a quadratic equation, the vertex is the **locator point**. It gives you a starting point for graphing the parabola quickly. The vertex for the quadratic equation $f(x) = a(x-h)^2 + k$ is the point (h, k). For $f(x) = -2(x+4)^2 + 7$ the vertex is $(-4, 7)$. Since $g(x) = 5(x-8)^2$ can also be written $g(x) = 5(x-8)^2 + 0$, the vertex is $(8, 0)$. We can rewrite $h(x) = \frac{3}{5}x^2 - \frac{2}{5}$ as $h(x) = \frac{3}{5}(x-0)^2 - \frac{2}{5}$ to see that its vertex is $\left(0, \frac{2}{5}\right)$.

Example 3

In a neighborhood water balloon battle, Dudley has developed a winning strategy. He has his home base situated five feet behind an eight-foot fence. 25 feet away on the other side of the fence is his nemesis' camp. Dudley uses a water balloon launcher, and shoots his balloons so that they just miss the fence and land in his opponent's camp. Write an equation that, when graphed, will model the trajectory (path) of the water balloon.





As with many problems, it is most helpful to first draw a sketch of the situation. The parabola shows the path the balloon will take, starting five feet away from the fence (point A) and landing 25 feet past the fence (point B).

There are different ways to set up axes for this problem, and depending where you put them, your answer might be different. Here, the y-axis will be at the fence. With the axes in place, we label any coordinates we know. This now shows all of the information we have from the problem description. If we can find the coordinates of the vertex (highest point) of this parabola, we will be able to write the equation of it in graphing form. Parabolas are symmetric, therefore the vertex will be half-way between the two x-intercepts. The total distance between points A and B is 30 units, so half is 15. Fifteen units from point A is the point $(10, 0)$. We know the equation will be in the form $y = a(x-10)^2 + k$, with $a < 0$. Also, k must be greater than eight since the vertex is higher than the y-intercept of $(0, 8)$. The parabola passes through the points $(0, 8)$, $(-5, 0)$ and $(25, 0)$. We will use these points in the equation we have so far and see what else we can find.
Using the point $(0, 8)$ we can substitute the x- and y-values in to the equation and write:

$$8 = a(0-10)^2 + k$$
or
$$8 = 100a + k$$

This equation has two variables, which means we need another (different) equation with a and k to be able to solve for them. Using the point $(-5, 0)$:

$$0 = a(-5-10)^2 + k$$
or
$$0 = 225a + k$$

Begin solving by subtracting the second equation from the first:

$$\begin{aligned} 8 &= 100a + k \\ -(0 &= 225a + k) \\ \hline 8 &= -125a \\ a &= -\tfrac{8}{125} \end{aligned}$$

Substitute this value for the variable a back into one of the two equations above to find k.

$$\begin{aligned} 8 &= 100\left(-\tfrac{8}{125}\right) + k \\ 8 &= -\tfrac{32}{5} + k \\ k &= \tfrac{72}{5} \end{aligned}$$

The equation for the path of a water balloon is $y = -\frac{8}{125}(x-10)^2 + \frac{72}{5}$. You should graph this on your graphing calculator to check.

Problems

answers follow

Find the x- and y-intercepts of each of the following quadratic equations.

1. $y = x^2 + 4x + 3$
2. $y = x^2 + 5x - 6$
3. $y = 2x^2 - 7x - 4$
4. $y = -3x^2 - 10x + 8$
5. $y = 16x^2 - 25$
6. $y = 6x - 12$

Find the error in each of the following solutions. Then find the correct solution to the problem.

7. Solve for x if $3x^2 + 6x + 1 = 0$.

$a = 3, b = 6, c = 1$

$$\begin{aligned} x &= \frac{6 \pm \sqrt{6^2 - 4(3)(1)}}{2(3)} \\ &= \frac{6 \pm \sqrt{36-12}}{6} \\ &= \frac{6 \pm \sqrt{24}}{6} \\ &= \frac{6 \pm 2\sqrt{6}}{6} \\ &= \frac{3 \pm \sqrt{6}}{3} \end{aligned}$$

8. Solve for x if $-2x^2 + 7x + 5 = 0$

$a = -2, b = 7, c = 5$

$$\begin{aligned} x &= \frac{-7 \pm \sqrt{7^2 - 4(-2)(5)}}{2(-2)} \\ &= \frac{-7 \pm \sqrt{49-40}}{-4} \\ &= \frac{-7 \pm 3}{-4} \end{aligned}$$

$$x = \frac{-4}{-4} = 1 \text{ or } x = \frac{-10}{-4} = 2.5$$

Find the vertex of each of the following parabolas by "averaging the x-intercepts" or "completing the square." Then write each equation in graphing form, and sketch the graph.

9. $y=-2x^2+4x+1$

10. $y=x^2+10x+19$

11. $y=(x+7)(x-3)$

12. $y=2(x+6)^2-1$

For each situation, write an appropriate equation that will model the situation effectively.

13. When Twinkle Toes Tony kicked a football, it landed 100 feet from where he kicked it. It also reached a height of 125 feet. Write an equation that, when graphed, will model the path of the ball from the moment it was kicked until it first touched the ground.

14. When some software companies develop software, they do it with "planned obsolescence" in mind. This means that they plan on the sale of the software to rise, hit a point of maximum sales, then drop and eventually stop when they release a newer version of the software. Suppose the curve showing the number of sales over time is parabolic and that the company plans on the "life span" of its product to be six months, with maximum sales reaching 1.5 million units. Write an equation that best fits this data.

15. A new skateboarder's ramp just arrived at Bungey's Family Fun Center. A cross-sectional view shows that the shape is parabolic. The sides are 12 feet high and 15 feet apart. Write an equation that, when graphed, will show the cross section of this ramp.

Answers

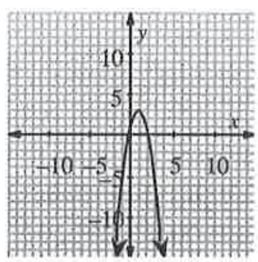
here

1. x-intercepts: $(-1, 0)$, $(-3, 0)$, y-intercept: $(0, 3)$.
2. x-intercepts: $(-6, 0)$, $(1, 0)$, y-intercept: $(0, -6)$.
3. x-intercepts: $(-0.5, 0)$, $(4, 0)$, y-intercept: $(0, -4)$.
4. x-intercepts: $(-4, 0)$, $\left(\frac{2}{3}, 0\right)$, y-intercept: $(0, 8)$.
5. x-intercepts: $\left(-\frac{5}{4}, 0\right)$, $\left(\frac{5}{4}, 0\right)$, y-intercept: $(0, -25)$.
6. x-intercept: $(2, 0)$, y-intercept: $(0, -12)$.
7. The formula starts with "$-b$;" the negative sign is left off; $x = \frac{-3 \pm \sqrt{6}}{3}$.
8. Under the radical, "$-4ac$" should equal + 40. $x = \frac{-7 \pm \sqrt{89}}{-4}$.

9. $y = -2(x-1)^2 + 3$

   x-intercepts: $\left(\frac{2 \mp \sqrt{6}}{2}, 0\right)$

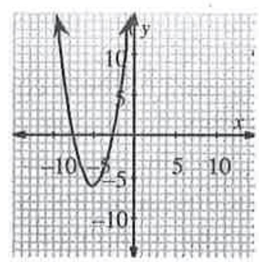
   vertex is $(1, 3)$



10. $y = (x+5)^2 - 6$

    x-intercepts: $\left(-5 \pm \sqrt{6}, 0\right)$

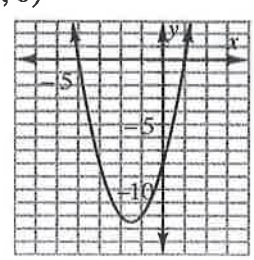
    vertex is $(-5, -6)$



11. $y = (x+2)^2 - 25$

    x-intercepts: $(3, 0)$, $(-7, 0)$

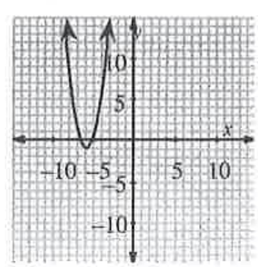
    vertex is $(-2, -25)$



12. $y = 2(x+6)^2 - 1$

    x-intercepts: $\left(\frac{-12 \pm \sqrt{2}}{2}, 0\right)$

    vertex is $(-6, -1)$



13. Placing the start of the kick at the origin gives $y = -0.05x(x-100)$.
14. Let the x-axis be the number of months, and the y-axis be the number of sales in millions. Placing the origin at the beginning of sales, we can use $y = -\frac{1}{6}(x-3)^2 + 1.5$.
15. Placing the lowest point of the ramp at the origin gives $y = \frac{4}{75}x^2$.

TRANSFORMING PARENT GRAPHS 2.2.1 and 2.2.3

Students will generalize what they have learned about transforming the graph of $f(x)=x^2$ to change the shape and position of the graphs of several other functions. The students start with the simplest form of each function's graph, which is called the "parent graph." Students use $y=x^3$, $y=\frac{1}{x}$, $y=\sqrt{x}$, and $y=b^x$ as the equations for parent graphs, and what they learn while studying these graphs will apply to all functions. They also learn to apply their knowledge to non-functions. For further information see the Math Notes boxes in Lessons 2.2.2 and 2.2.3.

Example 1

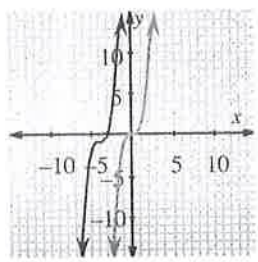
study

For each of the following equations, state the parent equation, and use it to graph each equation as a transformation of its parent equation.

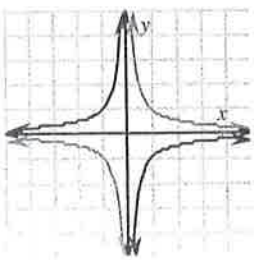
$$y=(x+4)^3-1 \qquad y=-\frac{1}{x}$$

$$y=3\sqrt{x-2} \qquad y=3^x-6$$

For each of these equations, we will graph both it and its parent on the same set of axes to help display the change and movement.

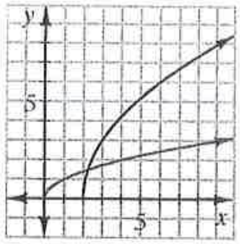


The first equation is a cubic (the term given to a polynomial with 3 as the highest power of x), thus its parent is $y=x^3$. The given equation will have the same shape as $y=x^3$, but it will be shifted to the left 4 units (from the "+ 4" within the parentheses), and down one unit (from the "–1"). The new graph is the darker curve shown on the graph at right. Notice that the point $(0,0)$ on the original graph has shifted left 4, and down 1, and now is at $(-4,-1)$. This point is known as a **locator point**. It is a key point of the graph, and graphing its position helps us to graph the rest of the curve.

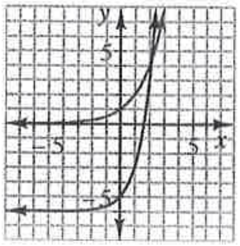


The second curve, $y=-\frac{1}{x}$, has had only one change from the parent graph $y=\frac{1}{x}$: the negative sign. Just as a negative at the front of $f(x)=x^2$ would flip this graph upside down, the negative sign here "flips" the parts of the parent graph across the x-axis. The lighter graph shown at right is the parent $y=\frac{1}{x}$, and the darker graph is $y=-\frac{1}{x}$.

In the equation $y = 3\sqrt{x-2}$ the radical is multiplied by 3, hence the transformed graph will grow vertically more quickly than the parent graph $y = \sqrt{x}$. It is also shifted to the right 2 units because of the "-2" under the radical sign. The new graph is the darker curve on the graph shown at right. Notice that the point (0, 0) on the original graph (the locator point) has shifted right 2 units.



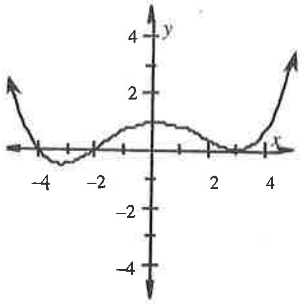
This last graph is an exponential function. The parent graph, $y = b^x$, changes in steepness as b changes. The larger b is, the quicker the graph rises, making a steeper graph. With $y = 3^x - 6$, the graph is a bit steeper than $y = 2^x$, often thought of as the simplest exponential function, but also shifted down 6 units. The lighter graph is $y = 2^x$, while the darker graph is $y = 3^x - 6$.



Example 2

Study this

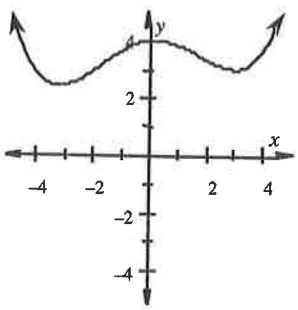
Suppose $f(x)$ is shown at right. From all you have learned about changing the graphs of functions:



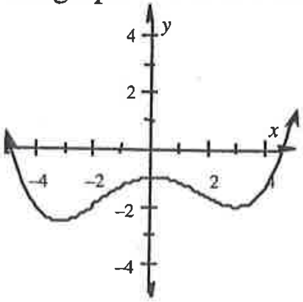
a. Graph $f(x)+3$. b. Graph $f(x)-2$. c. Graph $f(x-1)$.

d. Graph $f(x+3)$. e. Graph $3f(x)$. f. Graph $\frac{1}{2}f(x)$.

Each time we alter the equation slightly, the graph is changed. Even though we have no idea what the equation of this function is, we can still shift it on the coordinate grid. Remember that $f(x)$ represents the range or y-values. Therefore, in part (a), $f(x)+3$ says "the y-values, plus 3." Adding three to all the y-values will shift the graph up three units. This is shown at right. Notice that the shape of the graph is identical to the original, just shifted up three units. Check this by comparing the y-intercepts.

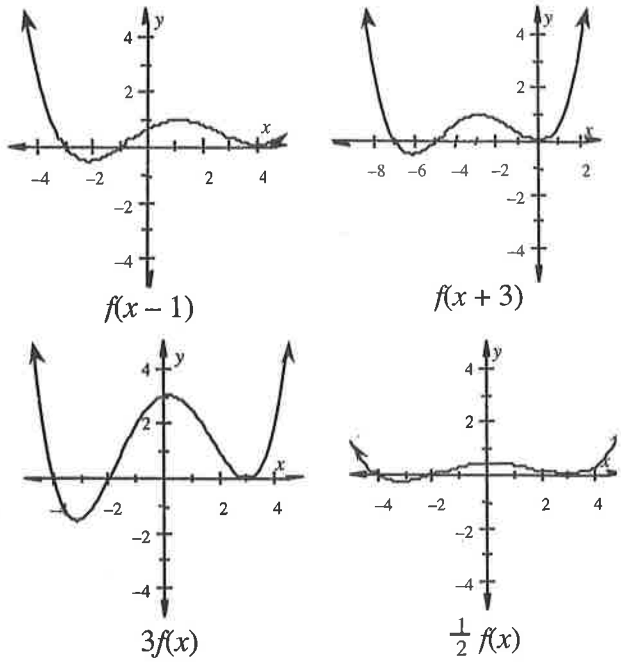


If $f(x)+3$ shifts the graph up three units, then $f(x)-2$ will shift the graph down two units. This graph is shown at left below.



Again, compare the y-intercept on the graph at left to the original. (Note: Using the y-intercept or the x-intercepts to help you graph is an effective way to create a graph.)

What happens when the change is made within the parentheses as in parts (c) and (d)? Here the shift is with the x-coordinates, thus the graph will move left or right. In part (c), the graph of $f(x-1)$ is shifted to the **right** 1 unit. The graph in part (d), $f(x+3)$, is shifted to the **left** 3 units. Both graphs are shown at right.



When multiplying $f(x)$ by a number as in parts (e) and (f), look at some key points. In particular, consider the x-intercepts. Since the y-value is zero at these points, multiplying by any number will not change the y-value. Therefore, the x-intercepts do not change at all, but the y-intercept will. In the original graph, the y-intercept is 1, so $f(0)=1$. Multiplying by 3 will raise that point three times as high, to the point $(0,3)$. Multiplying by $\frac{1}{2}$ changes the y-intercept to $\left(0, \frac{1}{2}\right)$.

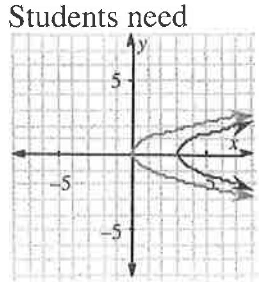
The larger the constant by which you multiply, the more stretched out the graph becomes. A smaller number flattens the graph.

Example 3

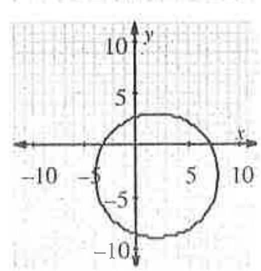
Apply your knowledge of parent graphs and transformations to graph the following two non-functions.

a. $x = y^2 + 3$ b. $(x-2)^2 + (y+3)^2 = 36$

Not every equation is a function, and the two non-functions students consider are $x = y^2$ and $x^2 + y^2 = r^2$. The first is the equation of a "sleeping parabola," or a parabola lying on its side. The second equation is the general form of a circle with center $(0,0)$, and radius of length r. As written, neither of these equations can be entered into a graphing calculator. Students need to solve each of these as "$y =$" to use the calculator. But rather than doing that, the students can use what they have already learned to make accurate graphs of each equation. The parent of the equation in part (a) is $x = y^2$. The "+ 3" tells us the graph will shift 3 units, but is it up, down, left, or right? Rewriting the equation as $\pm\sqrt{x-3} = y$ helps us see that this graph is shifted to the right 3 units. At right, the grey curve is the graph of $x = y^2$, and the darker curve is the graph of $x = y^2 + 3$.



Graphing the equation of a circle is straightforward: a circle with center (h, k) and radius r has the equation $(x-h)^2 + (y+k)^2 = r^2$. Therefore the graph of the equation in part (b) is a circle with a center at $(2, -3)$, and a radius of 6. The graph of the circle is shown at right.



answers to follow

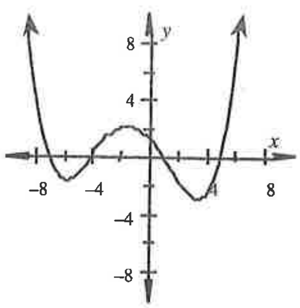
Problems

For each of the following equations, state the parent equation and then sketch its graph. Be sure to include any key and/or locator points.

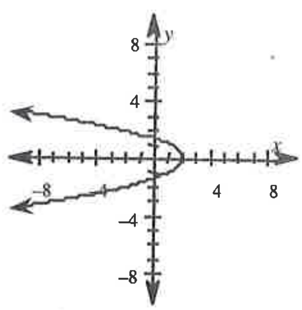
1. $y=(x-5)^2$
2. $y=-\frac{1}{3}(x+4)^2+7$
3. $(x-2)^2+(y+1)^2=9$
4. $y=|x+5|-2$
5. $y=\frac{1}{x+1}+10$
6. $y=2^x-8$
7. $y=-(x-2)^3+1$
8. $y=\sqrt{x+7}$
9. $y=3|x-5|$
10. $y=\pm\sqrt{x-9}$

For each of the following problems, state whether or not it is a function. If it is not a function, explain why not.

11.



12.

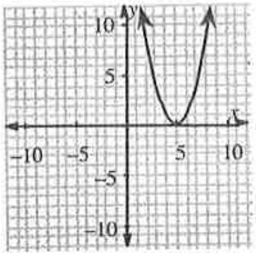


13. $y=7\pm\sqrt{9-x^2}$

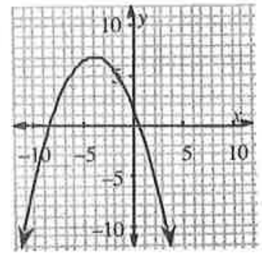
14. $y=3(x-4)^2$

Answers

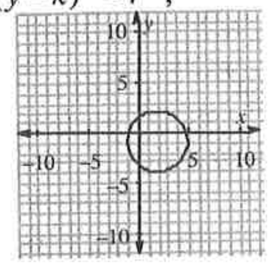
1. Parent graph $f(x)=x^2$, vertex $(5, 0)$



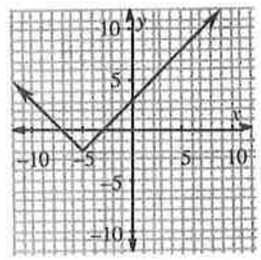
2. Parent graph $f(x)=x^2$, vertex $(-4, 7)$



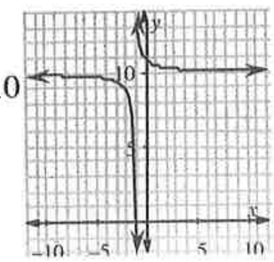
3. Parent graph $(x-h)^2+(y-k)^2=r^2$, center $(2, -1)$, radius 3



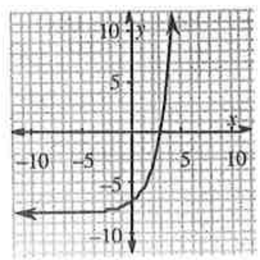
4. Parent graph $f(x)=|x|$, vertex $(-5, -2)$



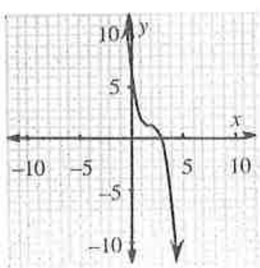
5. Parent graph $f(x)=\frac{1}{x}$, asymptotes $x=-1$, $y=10$



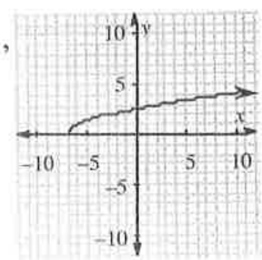
6. Parent graph $f(x)=2^x$, asymptote $x=-8$



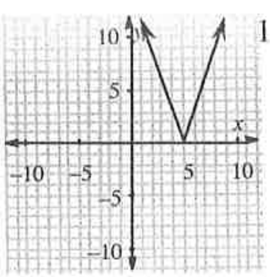
7. Parent graph $f(x)=x^3$, locator point $(2, 1)$



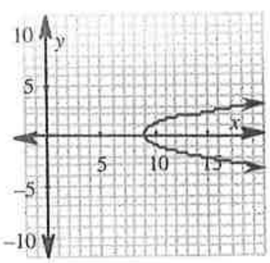
8. Parent graph $f(x)=\sqrt{x}$, vertex $(-7, 0)$



9. Parent graph $f(x)=|x|$, vertex $(5, 0)$



10. Parent graph $y^2=x$, vertex $(9, 0)$



11. Yes.

12. No, on the left part of the graph, for each x-value there are two possible y-values. You can see this by drawing a vertical line through the graph. If a vertical line passes through the graph more than once, it is not a function.

13. No, because the equation has "±," for each value substituted for x, there will be two y-values produced. A function can have only one output for each input.

14. Yes.

MORE ON COMPLETING THE SQUARE

Although students can find the vertex of a parabola by averaging the x-intercepts, they also can use the algebraic method known as completing the square. This allows students to go directly from standard (or non-graphing) form to graphing form without the intermediate step of finding the x-intercepts. Completing the square is also used when the equation of a circle is written in an expanded form. When the students first looked at how to complete the square, they used tiles so that they could see how the method works. When they tried to create a square (complete it) by arranging the tiles, there were either too many or missing parts. This visual representation helps students see how to rewrite the equation algebraically.

The Box method shown in class accomplishes the same thing.

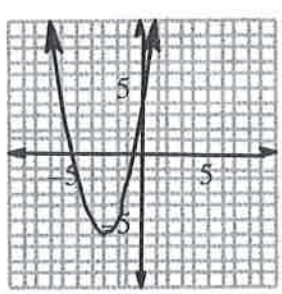
Example 1

The function $f(x) = x^2 + 6x + 3$ is written in standard form. Complete the square to write it in graphing form. Then state the vertex of the parabola and sketch the graph.

The general equation of a parabola in graphing form is $f(x) = a(x-h)^2 + k$, where (h, k) is the vertex. The original equation needs to be changed into a set of parentheses squared, with a constant either added to or subtracted from it. To do this, we must know that $(x-h)^2 = x^2 - 2xh + h^2$. We will use this form of a perfect square to complete the square of the given equation or function.

$$\begin{aligned} f(x) &= x^2 + 6x + 3 \\ &= x^2 + 6x + \boxed{} + 3 - \boxed{} \\ &= x^2 + 6x + \boxed{9} + 3 - \boxed{9} \\ &= (x+3)^2 - 6 \end{aligned}$$

The first box holds a space for the number we have to add to complete the square. The second box is to subtract that same number so as not to change the balance of the equation. To determine the missing number, take half the coefficient of x (half of 6), and then square it and place the result in both boxes. With the equation in graphing form, we know the vertex is $(-3, -6)$. The graph is shown below.



Example 2

The equation $x^2 - 8x + y^2 + 16y = 41$ is the equation of a circle. Complete the square to determine the coordinates of its center and the length of the radius.

As with the last example, we will fill in the blanks to create perfect squares. We need to do this twice: once for x, and again for y.

$$x^2 - 8x + y^2 + 16y = 41$$

$$x^2 - 8x + \boxed{} - \boxed{} + y^2 + 16y + \boxed{} - \boxed{} = 41$$

$$x^2 - 8x + \boxed{16} - \boxed{16} + y^2 + 16x + \boxed{64} - \boxed{64} = 41$$

$$(x-4)^2 - 16 + (y+8)^2 - 64 = 41$$

$$(x-4)^2 + (y+8)^2 = 41 + 16 + 64$$

$$(x-4)^2 + (y+8)^2 = 121$$

This is a circle with center $(4, -8)$ and a radius of $\sqrt{121} = 11$.

Problems

answers to follow

Write each of the following equations in graphing form. Then state the vertex and the direction the parabola opens.

1. $y = x^2 - 8x + 18$
2. $y = -x^2 - 2x - 7$
3. $y = 3x^2 - 24x + 42$
4. $y = 2x^2 - 6$
5. $y = \frac{1}{2}x^2 - 3x + \frac{1}{2}$
6. $y = x^2 + 18x + 97$

Find the center and radius of each circle.

7. $(x+2)^2 + (y+7)^2 = 25$
8. $3(x-9)^2 + 3(y+1)^2 = 12$
9. $x^2 + 6x + y^2 = 91$
10. $x^2 - 10x + y^2 + 14y = -58$
11. $x^2 + 50x + y^2 - 2y = -602$
12. $x^2 + y^2 - 8x - 16y = 496$

Answers

1. $y = (x-4)^2 + 2$, vertex $(4, 2)$, up
2. $y = -(x+1)^2 - 6$, vertex $(-1, -6)$, down
3. $y = 3(x-4)^2 - 6$, vertex $(4, -6)$, up
4. $y = 2(x-0)^2 - 6$, vertex $(0, -6)$, up
5. $y = \frac{1}{2}(x-3)^2 - 4$, vertex $(3, -4)$, up
6. $y = (x+9)^2 + 16$, vertex $(-9, -16)$, up
7. Center: $(-2, -7)$, radius: 5
8. Center: $(9, -1)$, radius: 2
9. Center: $(-3, 0)$, radius: 10
10. Center: $(5, -7)$, radius: 2
11. Center: $(-25, 1)$, radius: $\sqrt{24} = 2\sqrt{6}$
12. Center: $(4, 8)$, radius: 24